Gradient Operator in General Coordinates

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(Dated: December 27, 2009)

The gradient operator is expressed in general coordinates and then simplified for the case of orthogonal coordinates. The generalized gradient operator is then expressed in Schwarzschild coordinates and rotating coordinates.

I. GENERAL COORDINATES

When working in general coordinates, it is helpful to have the gradient operator already worked out ahead of time. In general coordinates, the gradient of a scalar function ψ is

$$\nabla\psi = g^{ik}\partial_k\psi\mathbf{g}_i, \tag{1}$$

where ∂_k denotes partial differentiation with respect to the general coordinate q^k, and $\mathbf{g}_i$ is a basis vector pointing in the $i-$coordinate direction. Writing out Eq. (1) in full gives

$$\begin{aligned}\nabla\psi =& \mathbf{g}_1\left(g^{11}\partial_1\psi + g^{12}\partial_2\psi + g^{13}\partial_3\psi\right) + \\ & \mathbf{g}_2\left(g^{21}\partial_1\psi + g^{22}\partial_2\psi + g^{23}\partial_3\psi\right) + \\ & \mathbf{g}_3\left(g^{31}\partial_1\psi + g^{32}\partial_2\psi + g^{33}\partial_3\psi\right).\end{aligned} \tag{2}$$

For the special case when $g_{\mu\nu}$ is diagonal, Eq. (2) simplifes greatly, leaving us with

$$\nabla\psi = \mathbf{g}_1 g^{11}\partial_1\psi + \mathbf{g}_2 g^{22}\partial_2\psi + \mathbf{g}_3 g^{33}\partial_3\psi. \tag{3}$$

Moreover, for the case of diagonal $g_{\mu\nu}$, we have the following relations

$$g^{ii} = 1/g_{ii} = \frac{1}{\sqrt{-g_{ii}}}\frac{1}{\sqrt{-g_{ii}}}. \tag{4}$$

Using Eq. (4) in Eq. (3) gives

$$\nabla\psi = \frac{\mathbf{e}_1}{\sqrt{-g_{11}}}\partial_1\psi + \frac{\mathbf{e}_2}{\sqrt{-g_{22}}}\partial_2\psi + \frac{\mathbf{e}_3}{\sqrt{-g_{33}}}\partial_3\psi. \tag{5}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are unit vectors pointing in their respective coordinate directions. Upon using the Lamé coefficients, $h_i = \sqrt{-g_{ii}}$, we arrive at the familiar form of the gradient in orthogonal coordinates:

$$\nabla\psi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\psi}{\partial q^1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\psi}{\partial q^2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\psi}{\partial q^3}, \tag{6}$$

Thus, when the metric tensor is diagonal, we can use Eq. (6), but when the coordinate system is non-orthogonal, we must resort to using the full expression of the gradient, given by Eq. (2).

With suitable expressions of the gradient in hand, we are ready to consider specific coordinate systems. In Schwarzschild coordinates, the metric tensor and inverse metric tensor are given by

$$g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\phi}{c^2} & 0 & 0 & 0 \\ 0 & -\left(1+\frac{2\phi}{c^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \tag{7a}$$

$$g^{\mu\nu} = \begin{pmatrix} \left(1+\frac{2\phi}{c^2}\right)^{-1} & 0 & 0 & 0 \\ 0 & -\left(1+\frac{2\phi}{c^2}\right) & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2}\sin^{-2}\theta \end{pmatrix}, \tag{7b}$$

where ϕ is the field potential of the gravitational source. Since the metric tensor is diagonal, we can use Eq. (6). Using Eq. (6) and the components of Eqs. (7), it is straightforward to see that the gradient operator assumes the form

$$\nabla \to \left(1+\frac{2\phi}{c^2}\right)^{1/2}\mathbf{e}_r\frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\mathbf{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{8}$$

where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ are respective unit vectors in the (r, θ, ϕ) directions.

Another frequently encounted coordinate system is the rotating coordinate system. In rotating coordinates, the metric and inverse tensors are

$$g_{\mu\nu} = \begin{pmatrix} 1-v^2 & -v_x & -v_y & 0 \\ -v_x & -1 & 0 & 0 \\ -v_y & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{9a}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & -v_x & -v_y & 0 \\ -v_x & -1+v_x^2 & v_x v_y & 0 \\ -v_y & v_x v_y & -1+v_y^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{9b}$$

wherein $v_x = -\omega y$, $v_y = \omega x$, $v^2 = v_x^2 + v_y^2$, and we have put $c = 1$ for the sake of simplicity. As can be seen in Eqs. (9), the metric tenor is not diagonal in rotating coordinates. Thus, Eq. (6) cannot be used; rather, we must resort to using Eq. (2). Expressed in terms of the metric components, the gradient operator becomes

$$\begin{aligned}\nabla \to& \mathbf{g}_1\left(g^{11}\frac{\partial}{\partial x} + g^{12}\frac{\partial}{\partial y}\right) + \dots \\ & + \mathbf{g}_2\left(g^{21}\frac{\partial}{\partial x} + g^{22}\frac{\partial}{\partial y}\right) + \mathbf{g}_3 g^{33}\frac{\partial}{\partial z}.\end{aligned} \tag{10}$$

Using the components of the inverse metric tensor, Eq. (9b), and simplifying a bit, leads to

$$-\nabla \to \left(\mathbf{e}_x \left(1 - \omega^2 y^2\right) + \mathbf{e}_y \omega^2 xy\right) \frac{\partial}{\partial x} + \dots \\ + \left(\mathbf{e}_y \left(1 - \omega^2 x^2\right) + \mathbf{e}_x \omega^2 xy\right) \frac{\partial}{\partial y} + \dots \\ + \mathbf{e}_z g^{33} \frac{\partial}{\partial z}, \tag{11}$$

where $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ are unit vectors pointing in the (x, y, z) coordinate directions, respectively.