

Gravitation in Material Media

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Abstract. When two gravitating bodies reside in a material medium, Newton's law of universal gravitation must be modified to account for the presence of the medium. A modified expression of Newton's law is known in the literature, but lacks a clear connection with existing gravitational theory. Newton's law in the presence of a homogeneous material medium is herein derived on the basis of classical, Newtonian gravitational theory and by a general relativistic use of Archimedes' principle. It is envisioned that the techniques presented herein will be most useful to graduate students and those undergraduate students having prior experience with vector analysis and potential theory.

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1. Introduction

Certainly one of the most famous of all equations in physics is Newton's law of universal gravitation, given by

$$\mathbf{F} = -G \frac{m_A m_B}{r^2} \mathbf{e}_r \quad (1)$$

where \mathbf{F} is the vector force acting between two neighboring bodies, m_A and m_B are the masses of the bodies, r is the distance between the centers of the bodies, and \mathbf{e}_r is a unit vector pointing in the r -coordinate direction. The minus sign indicates that the gravitational force is attractive, and G is the familiar gravitational constant. Virtually every undergraduate text in existence discusses (1) at length. What doesn't seem to be as widely discussed, if at all, is that when the two bodies reside within a material medium, such as water, the force between the bodies is affected by the presence of the medium.

According to Z. Horák [1], when a material medium is taken into account, (1) assumes the form

$$\mathbf{F} = -G \frac{m_A m_B}{r^2} \left(1 - \frac{\rho_f}{\rho_A}\right) \left(1 - \frac{\rho_f}{\rho_B}\right) \mathbf{e}_r \quad (2)$$

in which ρ_A and ρ_B are the mass-densities of the gravitating bodies, and ρ_f is the mass-density of the medium. Equation (2) appears to be little discussed in the literature. One exception is B. Vybíral [2], who reported several Cavendish-type experiments performed with large metal spheres submerged in water, rather than air, leading to very close agreement with (2). Indeed, Z. Horak [1] provides a derivation of (2), but unfortunately it seems loosely connected with conventional theory. What is missing is a derivation of (2) that clearly relies on accepted theory and is digestible at the undergraduate level. Considering that (2) is so little discussed in the literature, it is anticipated that the techniques presented in the following sections will be useful in the teaching of gravitation and potential theory, as well as providing students with some tools necessary for tackling problems beyond the simplest, most popular cases.

In the next section, techniques for deriving (2) are demonstrated on the basis of classical, Newtonian gravitational theory. The section begins with a direct application of the gravitational field potential [3] to the medium, leading to divergent results due to the extent of the medium. As a solution to this problem, it is pointed out that according to Gauss' law, a uniformly distributed material medium makes no "net" contribution to the gravitational potential [4, 5]. Based on this, the gravitational potential is then normalized by subtracting the mass-density of the medium everywhere, including from the two gravitating bodies, thus ensuring that the potential goes to zero in the far-field limit. Working under these conditions leads directly to an expression of the gravitational force identical to (2).

In section 3, a generalized form of Archimedes' principle [6] is used, in which the buoyancy force on one body is expressed in terms of a combined metric tensor due to the other body and the material medium [7]. Using Archimedes' principle and the combined metric tensor leads directly to an expression of the force in agreement with (2).

2. Classical Considerations

A standard way to derive the gravitational force between two bodies is to first derive an expression for the gravitational field potential ϕ due to one body and then use the familiar expression $\mathbf{F} = -m\nabla\phi$ to find the force on the second body. In absence of other outside influences, Newton's third law of motion assures that the force on the two bodies is mutual.

Let us consider two spherical bodies, called A and B , having respective mass-densities ρ_A and ρ_B , residing within a homogeneous material medium, having a mass-density ρ_f . The centers of the bodies are separated by a distance r . Moreover, suppose the two bodies and the medium are in thermodynamic equilibrium so as to eliminate any phenomena arising due to temperature differences. The gravitational field potential of the first body, say body A , can be computed by use of the familiar expression [3]

$$\phi = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \quad (3)$$

where $\rho(\mathbf{x}')$ is the mass-density of the gravitational source, \mathbf{x} is a vector pointing from the coordinate origin to a point of observation, \mathbf{x}' is a vector pointing from the origin to an elementary portion of the gravitational source, and $|\mathbf{x} - \mathbf{x}'|$ is the magnitude of the vector pointing from the elementary source to the point of observation. Applying (3) to the region exterior to body A , and carrying out the angular integrations, leaves us with

$$\phi_A = -\frac{4\pi G}{r} \left(\int_0^a \rho_A r'^2 dr' + \int_a^r \rho_f r'^2 dr' \right) - 4\pi G \int_r^\infty \rho_f r' dr' \quad (4)$$

in which a is the radius of body A , and the subscript on ϕ_A indicates that the potential is due to body A . Clearly, the right-most integral diverges as a consequence of the extent of the medium. At first sight, this seems a bit confusing. All sources of gravitation ought to be included in (3) and (4); but at the same time, we should expect the gravitational potential to be finite everywhere.

A way around the divergence of (4) reveals itself upon considering Gauss' law, expressed for gravitation as [4, 5]

$$\oint_S \mathbf{g} \cdot \mathbf{n} da = -4\pi G \int_V \rho(\mathbf{x}) d^3x. \quad (5)$$

The integral on the right-hand side of (5) expresses the mass contained within an enclosed volume, V , while the left-hand side expresses the gravitational flux passing through the exterior surface, S , of V . The point to notice is that the gravitational flux is proportional to the "net" mass enclosed within the surface [4]. Since the material medium is uniformly distributed everywhere, the medium makes no net contribution to (4). This suggests that we can normalize (3) and (4) so that $\phi \rightarrow 0$ in the limit of far fields by essentially subtracting the mass-density of the medium everywhere, including from bodies A and B . Doing this sets $\rho_f = 0$ outside bodies A and B and puts the mass-densities of bodies A and B in the form

$$\rho_A^* = \rho_A - \rho_f \quad (6)$$

$$\rho_B^* = \rho_B - \rho_f. \quad (7)$$

Returning to (4) and putting $\rho_f = 0$ eliminates the two right-most integrals, while using (6) for the mass-density of body A leads directly to

$$\phi_A = -G \frac{m_A^*}{r} \quad (8)$$

in which the mass of body A is $m_A^* = \rho_A^* V_A$.

With an expression for the gravitational potential due to body A in hand, we can determine the force on body B by simply taking the gradient of (8) and then using $\mathbf{F}_B = -m_B^* \nabla \phi_A$. Carrying this out gives the force on body B as

$$\mathbf{F}_B = -G \frac{m_A^* m_B^*}{r^2} \mathbf{e}_r. \quad (9)$$

Equation (9) has the form of (1) with the exception that m_A^* and m_B^* are expressed in terms of (6) and (7), respectively. Using (6) and (7), and rearranging a bit puts (9) in a form equivalent to (2):

$$\mathbf{F} = -G \frac{m_A m_B}{r^2} \left(1 - \frac{\rho_f}{\rho_A}\right) \left(1 - \frac{\rho_f}{\rho_B}\right) \mathbf{e}_r \quad (10)$$

in which the subscript has been dropped from \mathbf{F}_B in recognition that (10) is mutual between bodies A and B .

An equivalent approach to deriving (10) is to solve the gravitational Poisson's equation, $\nabla^2 \phi = 4\pi G \rho_f$, for ϕ in the region outside bodies A and B where ρ_f is the mass-density of the material medium. Carrying this out in spherical coordinates leads to the expressions

$$\frac{d\phi}{dr} = \frac{4\pi}{3} G \rho_f r + \frac{c_1}{r^2} \quad (11)$$

$$\phi = \frac{4\pi}{6} G \rho_f r^2 - \frac{c_1}{r} + c_2 \quad (12)$$

where c_1 and c_2 are constants of integration. Clearly, (11) and (12) diverge as $r \rightarrow \infty$. A way around the divergence, however, is to put $\nabla^2 \phi_f = 4\pi G \rho_f$, in which ϕ_f is the field potential due strictly to the material medium, and ϕ is understood to be the potential due solely to the matter comprising the gravitational source. Carrying this out in Poisson's equation leads to the simpler, Laplacian expression

$$\nabla^2 \phi_N = 0 \quad (13)$$

where $\phi_N = \phi - \phi_f$ and the subscript on ϕ_N indicates a "net" field potential. It should be understood that the steps taken with respect to (13) are equivalent to introducing (6) and (7) for use in (3). It is straightforward to show that (13) leads to the expressions

$$\frac{d\phi_N}{dr} = \frac{c_1}{r^2} \quad (14)$$

$$\phi_N = -\frac{c_1}{r} + c_2 \quad (15)$$

where, as above, c_1 and c_2 are constants of integration.

The next order of business is to determine expressions for c_1 and c_2 . Let us suppose that body A is the gravitational source. Upon inspecting (15), it is easy to see that in order for $\phi_N \rightarrow 0$ as $r \rightarrow \infty$, we must put $c_2 = 0$. Thus, c_2 drops out of (15). To determine c_1 , recall that body A is a solid sphere having a radius $r = a$. Using the expression $\phi_N = \phi - \phi_f$ in (14), and then evaluating at $r = a$ gives

$$\frac{c_1}{a^2} = \frac{d\phi(a)}{dr} - \frac{d\phi_f(a)}{dr} \quad (16)$$

Comparing (16) and (11) makes it clear that the contribution due to the medium at $r = a$ is

$$\frac{d\phi_f(a)}{dr} = \frac{4\pi}{3}G\rho_f a \quad (17)$$

Moreover, since body A is a solid sphere, and thus the material medium does not reside within the region $r \leq a$, we should expect that

$$\frac{d\phi(a)}{dr} = \frac{4\pi}{3}G\rho_A a \quad (18)$$

where ρ_A is the mass-density of body A . Using (17) and (18) in (16) and then substituting into (15) leads to

$$\phi_N = -\frac{4\pi}{3r}Ga^3(\rho_A - \rho_f) \quad (19)$$

As a final consideration, upon noting that the total mass of body A is $m_A = \rho_A V_A$, (19) can be recast in the form

$$\phi_N = -G\frac{m_A}{r} \left(1 - \frac{\rho_f}{\rho_A}\right) \quad (20)$$

It will be recognized that (20) is equivalent to (8). As discussed with respect to (8), the force on a nearby body, B , can be determined by taking the gradient of (20) and then using the familiar expression $\mathbf{F}_B = -m_B^* \nabla \phi_N$. Carrying this out leads directly to an expression of the force identical to (10).

3. Archimedes' Principle

Inspection of (2) suggests that each body is buoyed by the medium due to the gravitational field of the other body. Indeed, another way to derive Newton's law in the presence of a material medium is by using Archimedes' principle. Let us again consider the two bodies A and B residing in a uniformly distributed material medium and having a fixed distance r between the centers of the two bodies. As mentioned in the previous section, it is assumed that the two bodies and the medium are in thermodynamic equilibrium, so as to eliminate any phenomena due to differences in temperature.

So long as bodies A and B remain stationary, and their respective gravitational fields are time-independent, the vector force on each body is expressible as [6]

$$\mathbf{F} = \frac{mc^2}{2} \frac{g^{ij} g_{00,j}}{g_{00}} \left(1 - \frac{\rho_f}{\rho}\right) \mathbf{e}_i \quad (21)$$

in which g_{00} and g^{ij} are respective components of the metric tensor $g_{\mu\nu}$ and the inverse metric tensor $g^{\mu\nu}$, c is the speed of light, the comma denotes partial differentiation, and Latin indices are taken over the values (1, 2, 3). The vector \mathbf{e}_i is a basis vector pointing in the i -coordinate direction, and ρ is the mass-density of the body experiencing the force. Equation (21) is the apparent weight of each body due to the gravitational influence of the other body and the medium.

Upon applying (21) to one body, the metric tensor $g_{\mu\nu}$ should be understood as expressing the space-time curvature due to the combined influence of the other body and the intervening medium. Let us use (21) to determine the force on body A , due to the presence of body B and the medium. Were the two bodies situated in a vacuum, $g_{\mu\nu}$ due to body B would be expressed in Schwarzschild coordinates. With the medium present, however, there is an additional contribution to $g_{\mu\nu}$. The contribution of the medium can be determined by considering the case in which body B is absent, leaving an empty cavity in the medium. Within the medium, outside the cavity, g_{00} assumes the particularly simple form [7]

$$g_{00} = 1 + \frac{2Gm_f}{rc^2} \quad (22)$$

where m_f is the mass of the portion of the medium that would otherwise fill the cavity. With body B occupying the cavity, the superposition principle suggests that we may put

$$g_{00} = 1 - \frac{2G}{rc^2}(m_B - m_f). \quad (23)$$

It is to be noticed that (23) reduces to the familiar Schwarzschild form when $m_B \neq 0$ and $m_f \rightarrow 0$. Substituting (23) into (21), and putting $g^{11} = -1$ to limit things to first order terms, leads to

$$\mathbf{F} = -G \frac{m_A m_B}{r^2} \left(1 - \frac{\rho_f}{\rho_A}\right) \left(1 - \frac{\rho_f}{\rho_B}\right) \mathbf{e}_r. \quad (24)$$

As expected, (24) is identical to (2) and (10). Unlike the approach used in the previous section, however, (24) was derived solely by applying Archimedes' principle in the form of (21).

4. Discussion and Conclusions

As discussed in the Introduction, when two gravitating bodies reside in a material medium, Newton's law of universal gravitation must be modified to account for the presence of the medium. Such a modified expression of the force does indeed exist in the literature [1, 2], but appears to lack a clear connection with existing gravitational theory. Herein, Newton's law in the presence of a homogeneous material medium has been derived on the basis of classical, Newtonian gravitational theory and by a general relativistic use of Archimedes' principle [6]. In the former, a direct application of the gravitational field potential to the medium led to divergent results due to the extent of the medium. The solution to this problem hinged on noticing that according to

Gauss' law, a uniformly distributed material medium makes no "net" contribution to the gravitational potential. Rather, Gauss' law states that the net gravitational flux emanating from an enclosed surface is proportional to the net mass enclosed within the surface [4, 5]. Knowledge of this point justified normalizing the gravitational potential by subtracting the mass-density of the medium everywhere, including from the two gravitating bodies. Carrying this out ensured that the gravitational potential approaches zero in the far-field limit. Working under these conditions led directly to an expression of the gravitational force in agreement with the literature [1, 2].

In section 3, a generalized form of Archimedes' principle was used, in which the buoyancy force on one body was expressed in terms of the metric tensor due to the gravitation of the other body and the material medium [6]. In order to account for the presence of the medium, the metric tensor in Schwarzschild coordinates was combined with the metric tensor for an empty cavity in the medium [7]. Using the combined metric tensor led directly to the expected expression of the force [1, 2].

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