Forces in General Relativity

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Abstract. Many textbooks dealing with general relativity do not demonstrate the derivation of forces in enough detail. The analyses presented herein demonstrate straightforward methods for computing forces by way of general relativity. Covariant divergence of the stress-energy-momentum tensor is used to derive a general expression of the force experienced by an observer in general coordinates. The general force is then applied to the local co-moving coordinate system of a uniformly accelerating observer, leading to an expression of the inertial force experienced by the observer. Next, applying the general force in Schwarzschild coordinates is shown to lead to familiar expressions of the gravitational force. As a more complex demonstration, the general force is applied to an observer in Boyer-Lindquist coordinates near a rotating, Kerr black hole. It is then shown that when the angular momentum of the black hole goes to zero, the force on the observer reduces to the force on an observer held stationary in Schwarzschild coordinates. As a final consideration, the force on an observer moving in rotating coordinates is derived. Expressing the force in terms of Christoffel symbols in rotating coordinates leads to familiar expressions of the centrifugal and Coriolis forces on the observer. It is envisioned that the techniques presented herein will be most useful to graduate level students, as well as those undergraduate students having experience with general relativity and tensor analysis.

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1. Introduction

Essentially all textbooks dealing with general relativity provide students with a thorough understanding of the geometric aspects of the theory, as well as general tensor analysis techniques. While many texts touch on the subject of forces to varying degrees [1]-[3], what is often times lacking is a clear demonstration of how geometric objects such as the metric tensor and Christoffel symbols can be used to derive inertial and gravitational vector forces. The analyses presented herein demonstrate straightforward methods for computing forces by way of general relativity. Considering that this subject is covered only lightly in many textbooks, it is envisioned that the analyses presented in the following sections will be useful in the teaching of general relativity, as well as providing students with some of the tools necessary for computing forces on their own, and thereby help them gain a deeper understanding of general relativity.

An important point to recognize is that according to general relativity, gravitational forces are treated as manifestations of space-time geometry [1]-[9]. In the next section, an observer residing in arbitrary coordinates is used to demonstrate the relationship between forces and space-time geometry. It is pointed out that the coordinate system may be due to the observer's own acceleration, or due to the observer being stationed near a large gravitational source. Covariant divergence of the stress-energy-momentum tensor is used to derive a general expression of the force experienced by the observer. An alternative form of the general force is then derived by incorporating a lapse function, which is well known to express the quantity of proper time that elapses per unit of coordinate time [10].

Section 3 demonstrates the use of the general force and lapse function in familiar coordinate systems. The section begins with a discussion of the local co-moving coordinate system of a uniformly accelerating observer. Using the general force and the components of the metric tensor in the accelerating system leads quite simply to an expression of the inertial force experienced by the accelerating observer. Specializing to the case of weak acceleration produces an expression of the force equivalent to that obtainable by way of Newton's second law of motion. Next, Schwarzschild coordinates are considered. An expression of the lapse function for an observer held stationary near a large gravitational source is derived. Using the Schwarzschild lapse function with the general force of section 2 leads to a general relativistic expression of the gravitational force [10]. The familiar Newtonian expression of the force is retrieved upon specializing to the case of a weakly gravitating source.

The coordinate systems discussed in section 3 are simple, and consequently the derivations of the force are straightforward. In section 4, it is demonstrated that the general force is not limited solely to simple coordinate systems. The general force is applied to the Boyer-Lindquist coordinate system for an observer stationed outside a rotating, Kerr black hole. Since Boyer-Lindquist coordinates are so algebraically involved, the general force is tackled one component at a time, and the derivation is limited to key results. Once all the components of the force are obtained, the force on the observer is stated in totality. Section 4 closes with a demonstration that when the angular momentum of the source is put equal to zero, the force on the observer in Boyer-Lindquist coordinates reduces to the expression of the force on an observer held stationary in Schwarzschild coordinates.

In section 5, a form of the general force expressed in terms of Christoffel symbols is used for an observer moving in rotating coordinates. Expressing the Christoffel symbols in rotating coordinates leads to an expression of the force in terms of the total energy and momentum associated with the observer. Further simplification of the force leads to familiar expressions of the centrifugal and Coriolis forces on the observer.

In the appendix, the gradient operator is expressed in general coordinates. An expression of the gradient operator is then derived for the special case in which the metric tensor is diagonal. Upon introducing the Lamé coefficients, the gradient operator is shown to assume a form familiar to most students by way of vector analysis. The appendix closes with a simple derivation of the gradient operator in Schwarzschild coordinates.

2. Force in general coordinates

Let us consider an observer experiencing a force within an enclosed vessel. The force is such that the observer possesses weight and can stand at one end of the vessel. Ignoring tidal effects, and assuming the vessel is relatively small, such as an elevator car or telephone booth, the observer might conclude that the force is due to acceleration of the vessel or to gravitation. On one hand, the vessel may be suspended from a tether near a gravitational source. On the other hand, the vessel may be accelerating uniformly in Minkowski space-time under the action of an external force. In either case, the local reference frame of the observer will appear to be characterized by the effects of acceleration. With this model in mind, let us consider the force on the observer due to general coordinates in the frame of the observer.

According to general relativity, in the absence of any external forces, particles such as the above-mentioned observer move along geodesics, which are the straightest possible paths in space-time. One approach to deriving the force on the observer is to consider the motion of the observer by use of the geodesic equation, familiar to essentially all relativity textbooks. Another way in which the force can be derived is by use of the divergence of the stress-energy-momentum tensor, expressed in covariant form as

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{1}$$

where $T^{\mu\nu}$ is the stress-energy-momentum tensor of the observer, Greek indices are carried over the values $(0, 1, 2, 3)$, and the semi-colon (;) denotes covariant differentiation. Let us derive the force by way of (1) with the understanding that $T^{\mu\nu}$ includes all forms of energy and momentum, as well as any internal stress, associated with the observer.

Expanding the covariant derivative in (1) gives

$$\partial_\nu T^{\mu\nu} + \Gamma^\nu_{\alpha\nu} T^{\mu\alpha} + \Gamma^\mu_{\alpha\nu} T^{\alpha\nu} = 0 \tag{2}$$

in which ∂_ν denotes partial differentiation with respect to x^ν, and the Christoffel symbol $\Gamma^\mu_{\alpha\nu}$ is defined in terms of the metric tensor as

$$\Gamma^\mu_{\alpha\nu} = \frac{1}{2} g^{\mu\gamma} \left(g_{\alpha\gamma,\nu} + g_{\gamma\nu,\alpha} - g_{\alpha\nu,\gamma} \right). \tag{3}$$

Equation (2) can be simplified by expressing the Christoffel symbol in the second term in the form

$$\Gamma^\nu_{\alpha\nu} = \frac{\partial_\alpha(\sqrt{-g})}{(\sqrt{-g})} \tag{4}$$

where g is the determinant of the metric tensor, $g_{\mu\nu}$. Substituting (4) into the second term in (2), and rearranging a bit, gives

$$\frac{\partial_\nu(\sqrt{-g} T^{\mu\nu})}{(\sqrt{-g})} + \Gamma^\mu_{\alpha\nu} T^{\alpha\nu} = 0. \tag{5}$$

Summing over ν in the left-most term, and then integrating over the proper volume of the observer leads to

$$\int \frac{1}{c} \frac{dT^{\mu 0}}{d\tau} d^3x + \int \frac{\partial_i(\sqrt{-g} T^{\mu i})}{(\sqrt{-g})} d^3x + \int \Gamma^\mu_{\alpha\nu} T^{\alpha\nu} d^3x = 0 \tag{6}$$

in which τ is proper time, and Latin indices are taken over the values $(1, 2, 3)$. Equation (6) can be further simplified by converting the second term into a surface integral by way of Gauss' law. Carrying this out puts (6) in the form

$$\frac{1}{c} \frac{d}{d\tau} \int T^{\mu 0} d^3x + \oint T^{\mu i} dS_i + \Gamma^\mu_{\alpha\nu} \int T^{\alpha\nu} d^3x = 0 \tag{7}$$

where dS_i is an element of the proper, boundary surface surrounding the observer, and the time derivative has been moved outside the left-most integral because the volume of the observer is independent of time. It should also be noted that the Christoffel symbol has been moved outside the right-most integral on the assumption that the gradient of the gravitational potential is roughly uniform across the volume of the observer.

Upon applying (7) to the observer, it will be recognized that since the observer is standing on the floor of the vessel, and thus the vessel exerts a force on the observer, the stress-energy-momentum tensor of the vessel must be taken into account, as well. Let us express the stress-energy-momentum tensor of the observer-vessel system as

$$T^{\mu\nu} = T^{\mu\nu}_{(m)} + T^{\mu\nu}_{(v)} \tag{8}$$

where $T^{\mu\nu}_{(m)}$ and $T^{\mu\nu}_{(v)}$ are the stress-energy-momentum tensors of the observer and vessel, respectively. Upon substituting (8) into (7), it is straightforward to see that for $T^{\mu\nu}_{(v)}$ the first and third integrals are zero because the vessel is outside the boundary surface of the observer. For $T^{\mu\nu}_{(m)}$ the second integral is zero because the observer does not convey energy or momentum across the boundary surface of the observer. Under these conditions, (7) assumes the form

$$\frac{1}{c} \frac{d}{d\tau} \int T^{\mu 0}_{(m)} d^3x + \oint T^{\mu i}_{(v)} dS_i + \Gamma^\mu_{\alpha\nu} \int T^{\alpha\nu}_{(m)} d^3x = 0 \tag{9}$$

The left-most term in (9) can be expressed as a force f^μ on the observer due to the floor of the vessel and the coordinate system in the frame of the observer. The second term is an inward-directed, external force F^μ on the boundary surface of the observer by the floor of the vessel. Expressing (9) in terms of forces, and rearranging a bit, leads to [1]

$$f^\mu + \Gamma^\mu_{\alpha\nu} \int T^{\alpha\nu} d^3x = -F^\mu \tag{10}$$

where the subscript on $T^{\alpha\nu}_{(m)}$ has been dropped for brevity, and the minus sign on the right-hand side is included to express the force by which the observer acts on the floor of the vessel. Equation (10) is equivalent to the geodesic equation, with the exception that (10) is expressed in terms of the stress-energy-momentum tensor $T^{\alpha\nu}$ of the observer and includes the external force F^μ.

For the special case in which the observer remains stationary within the coordinate system, the momentum of the observer is zero, and thus $T^{\mu 0}_{(m)} = 0$ in (9) and $f^\mu = 0$ in (10). Moreover, when the metric tensor is independent of coordinate time, using (3) in (10) puts the force in the simplified form [11]

$$F^\mu = \frac{1}{2} g^{\mu j} g_{00,j} \int T^{00} d^3x. \tag{11}$$

The right-hand term can be further simplified by putting $T^{00} = \rho(U^0)^2$, using $g_{\mu\nu}U^\mu U^\nu = c^2$, and then performing the integration over the volume of the observer. Carrying this out and expressing the force as a vector leads to

$$\mathbf{F} = \frac{1}{2} E g^{ij} \partial_j \left(\ln(g_{00})\right) \mathbf{g}_i \tag{12}$$

in which $\mathbf{g}_i$ is a general basis vector pointing in the $i-$coordinate direction, and E is the total energy of the observer. It should be understood that the energy E encompasses all forms of energy associated with the observer, including those due to internal stresses and thermodynamic phenomena [12]-[14].

An expression equivalent to (12) can be obtained by noticing that the gradient operator can be expressed in general coordinates as [15]

$$\nabla \to -\mathbf{g}_i g^{ij} \partial_j \tag{13}$$

where the minus sign is included due to our choice of sign convention, $(+,\ -,\ -,\ -)$. Using (13) in (12), and rearranging a bit, puts the general force on the observer in the form

$$\mathbf{F} = E\nabla \left(\ln(g_{00})^{-\frac{1}{2}}\right). \tag{14}$$

As a further simplification, the observer's proper time can be related to coordinate time by [10]

$$g_{00} = \left(\frac{d\tau}{dt}\right)^2 \tag{15}$$

in which $d\tau$ is an interval of proper time, and dt is an interval of coordinate time. The function $d\tau/dt$ is known in the literature as a "lapse function," or "gravitational redshift factor," which expresses the quantity of proper time that elapses per unit of coordinate time [10]. Substituting (15) into (14) and simplifying leads to

$$\mathbf{F} = E\left(\frac{d\tau}{dt}\right)\nabla\left(\frac{dt}{d\tau}\right). \tag{16}$$

Equations (12) and (16) express the apparent weight of the observer acting on the floor of the vessel due to the general coordinate system. It should be understood that (12) and (16) are equivalent. Forces can be expressed in terms of g_{00}, or equivalently by use of the lapse function, $d\tau/dt$. For the sake of simplicity, the term "lapse function" is used casually in reference to $d\tau/dt$ as well as its inverse, $dt/d\tau$.

Referring specifically to (16), it is interesting to note that the force is solely dependent upon the total proper energy E of the observer and the gradient of the lapse function [10], [12]-[14]. The form of (16) clearly suggests that the lapse function behaves as a scalar potential, similar to the gravitational field potential, ϕ [10]. Unlike ϕ, however, the lapse function is purely relativistic in origin and is easily applicable to cases of acceleration, as well as gravitation. Perhaps, the best way to see the usefulness of (12) and (16) is to go ahead and consider specific coordinate systems.

3. Familiar coordinate systems

Let us return to the observer in the vessel mentioned in the previous section. Suppose the vessel is accelerating uniformly in Minkowski space-time due to an external force. In the co-moving frame of the accelerating observer, the components of the metric tensor can be expressed as [16, 17]

$$g_{00} = \left(1 + \frac{a_j x^j}{c^2}\right)^2 \tag{17a}$$

$$g_{ij} = -\delta_{ij} \tag{17b}$$

in which a_j is the proper acceleration, and x^j is the distance in flat space-time over which the acceleration takes place. The force experienced by the accelerating observer can be determined by use of either (12) or (16). Substituting (17a) into (12) leads to

$$\mathbf{F} = \frac{E a_j g^{ij}}{c^2}\left(1 + \frac{a_j x^j}{c^2}\right)^{-1}\mathbf{e}_i \tag{18}$$

in which $\mathbf{e}_i = \mathbf{g}_i\sqrt{-g^{ii}}$ is a basis vector of unit length pointing in the $i-$coordinate direction. Using (17b), it is easy to see that $g^{ij} = -\delta^{ij}$, and thus we can put $a^i = -\delta^{ij}a_j$. Moreover, upon noting that the acceleration vector of the observer is $\mathbf{a} = a^i\mathbf{e}_i$, and that $\mathbf{a}\cdot\mathbf{x} = a_j x^j$, (18) can be expressed equivalently as

$$\mathbf{F} = -m\mathbf{a}\left(1 + \frac{\mathbf{a}\cdot\mathbf{x}}{c^2}\right)^{-1} \tag{19}$$

in which the expression $E = mc^2$ has been used. It is straightforward to see that when $\mathbf{a}\cdot\mathbf{x} << c^2$, (19) reduces to

$$\mathbf{F} \approx -m\mathbf{a} \tag{20}$$

which is equivalent to the force obtainable by way of Newton's second law of motion. Equation (20) is the force by which the observer acts on the floor of the vessel, giving the observer a sense of possessing weight. It should be noted that the inertial mass m appearing in (19) and (20) is proportional to the total energy E associated with the observer, including sources of energy due to internal stresses and any thermodynamic phenomena that may be present [12]-[14].

Another interesting case is the Schwarzschild coordinate system. Suppose the vessel containing the observer is suspended from a tether near a large, spherically symmetric gravitational source of mass M. The observer has weight and can stand at the end of the vessel nearest the source. Let us use (16) to determine the force on the observer. The line element in Schwarzschild coordinates can be expressed as

$$ds^2 = c^2\left(1+\frac{2\phi}{c^2}\right)dt^2 - \left(1+\frac{2\phi}{c^2}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 \tag{21}$$

in which ϕ is the field potential due to the gravitational source. By inspection of (21), and using (15), it is straightforward to see that the lapse function in Schwarzschild coordinates is

$$\frac{d\tau}{dt} = \left(1+\frac{2\phi}{c^2}\right)^{\frac{1}{2}}. \tag{22}$$

Substituting (22) into (16) and using $E = mc^2$ to relate the gravitational mass of the observer to the total energy E associated with the observer leads to

$$\mathbf{F} = -m\left(1+\frac{2\phi}{c^2}\right)^{-1}\nabla\phi. \tag{23}$$

As a further simplification, (13) can be used to express the gradient operator in Schwarzschild coordinates. As shown in the appendix, expressing (13) in Schwarzschild coordinates puts the gradient operator in the form

$$\nabla \to \left(1+\frac{2\phi}{c^2}\right)^{1/2}\mathbf{e}_r\frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\mathbf{e}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}. \tag{24}$$

Using (24) in (23), and noting that outside the source the field potential is $\phi = -GM/r$, leads to [10]

$$\mathbf{F} = -\frac{GMm}{r^2}\left(1-\frac{2GM}{c^2r}\right)^{-\frac{1}{2}}\mathbf{e}_r. \tag{25}$$

Equation (25) is the weight of the observer acting on the floor of the vessel. When the source is weakly gravitating, limiting (23) and (25) to first order in ϕ puts the observer's weight in the more familiar form

$$\mathbf{F} \approx -m\nabla\phi = -\frac{GMm}{r^2}\mathbf{e}_r. \tag{26}$$

As mentioned above, the gravitational mass m in (26) is proportional to the total energy E of the observer.

An interesting aspect of the examples above is that the uniformly accelerating coordinate system is flat, despite the presence of the force on the accelerating observer.

Unlike the accelerating coordinate system, however, the Schwarzschild space-time is indeed curved. What the accelerating and Schwarzschild coordinate systems have in common is that in each case g_{00} is differentiable. In the Schwarzschild geometry, g_{00} is differentiable due to space-time curvature; whereas in the case of uniform acceleration, it is the motion of the observer that gives rise to a differentiable g_{00}. Thus, there is essentially no difference between gravitational and inertial forces; both ultimately hinge on g_{00}.

4. Boyer-Lindquist coordinates

The coordinate systems in the previous section are simple, and the computation of the forces straightforward. Now, let us consider a more complex coordinate system. An interesting coordinate system is Boyer-Lindquist coordinates, applicable to the geometry near a rotating, Kerr black hole. The line element in Boyer-Lindquist coordinates can be arranged into many seemingly different forms, though a particularly useful form for our purposes here is [10]

$$ds^2 = \frac{\rho^2}{\Sigma^2}\Delta dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\Sigma^2}{\rho^2}\sin^2\theta\left(d\varphi - \frac{2aMr}{\Sigma^2}dt\right)^2 \tag{27}$$

where the functions Δ, ρ^2, and Σ^2 are respectively given as

$$\Delta = r^2 + a^2 - 2Mr \tag{28a}$$

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{28b}$$

$$\Sigma^2 = \left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta \tag{28c}$$

in which $a = J/M$ is the angular momentum per unit mass, and we have temporarily put $GM/c^2 \to M$ for the sake of simplicity.

An interesting feature of the Kerr black hole is that all nearby objects are dragged into rotation in the $\varphi-$direction with an angular velocity $d\varphi/dt = 2aMr/\Sigma^2$ due to the black hole's rotation. Let us consider the force experienced by an observer that maintains fixed r and θ outside the Kerr hole, but is dragged along in the $\varphi-$direction.

One may question how (12) and (16) can be used to determine the force on an orbiting observer. An important point to notice is that we may view 3-dimensional space as being dragged along with the observer by the black hole. Thus, the observer remains at rest relative to 3-dimensional space, even though the observer appears to distant, stationary observers to be orbiting the Kerr black hole. For the interested reader, a detailed treatment of rotating black holes is given by K. Thorne, *et al* [10]. For our purposes, we simply point out that the orbiting observer remains at rest relative to 3-dimensional space so long as the observer maintains fixed r and θ, and the angular velocity is precisely $d\varphi/dt = 2aMr/\Sigma^2$. Thus, the force on the orbiting observer can indeed be determined by way of either (12) or (16).

Let us determine the force on the observer by using (12). For the case of the observer moving as described above, we may put $dr = d\theta = 0$ and $d\varphi = (2aMr/\Sigma^2)dt$ in (27). The resulting expression suggests that we then have

$$g_{00} = \frac{\rho^2}{\Sigma^2}\Delta. \tag{29}$$

Upon referring back to (28a), (28b), and (28c), it will be noted that (29) is independent of φ. Summing over i and j in (12) and retaining only non-zero terms then gives

$$\mathbf{F} = \frac{E}{2g_{00}}\left(g^{11}g_{00,1}\mathbf{g}_1 + g^{22}g_{00,2}\mathbf{g}_2\right). \tag{30}$$

Since working in Boyer-Lindquist coordinates is algebraically involved, it will be easier to work on each component of the force separately. Using (29) and starting with the partial differentiation in the first term on the right-hand side of (30), we may put

$$g_{00,1} = \frac{\rho^2}{\Sigma^2}\frac{\partial\Delta}{\partial r} + \frac{\Delta}{\Sigma^2}\frac{\partial\rho^2}{\partial r} + \rho^2\Delta\frac{\partial}{\partial r}\left(\frac{1}{\Sigma^2}\right). \tag{31}$$

Carrying out the partial differentiation of (28a), (28b), and (28c) with respect to r gives

$$\frac{\partial\Delta}{\partial r} = 2\left(r - M\right) \tag{32a}$$

$$\frac{\partial\rho^2}{\partial r} = 2r \tag{32b}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{\Sigma^2}\right) = \frac{-1}{\Sigma^4}\left[4r(r^2 + a^2) - 2ra^2\sin^2\theta + 2Ma^2\sin^2\theta\right]. \tag{32c}$$

Substituting (32a), (32b), and (32c) into (31), and then performing a lot of algebraic manipulation eventually leads to

$$g_{00,1} = -\frac{2}{\Sigma^4}\left[M\rho^2\left(r^4 - a^4\right) + 2a^2r^2M\Delta\sin^2\theta\right]. \tag{33}$$

Upon substituting (33) into (30), and noting that $g^{11} = -\Delta/\rho^2$ and $\mathbf{g}_1 = \mathbf{e}_r\rho/\sqrt{\Delta}$, the $r-$component of the force simplifies to

$$\mathbf{F}_r = -Gm\mathbf{e}_r\left[\frac{M\rho^2\left(r^4 - a^4\right) + 2a^2r^2M\Delta\sin^2\theta}{\rho^3\Sigma^2\sqrt{\Delta}}\right] \tag{34}$$

where we have put $M \to GM/c^2$ and have expressed the total gravitational mass of the observer by way of $E = mc^2$.

Turning now to the second term in (30), we have

$$g_{00,2} = \frac{\Delta}{\Sigma^2}\frac{\partial\rho^2}{\partial\theta} + \rho^2\Delta\frac{\partial}{\partial\theta}\left(\frac{1}{\Sigma^2}\right). \tag{35}$$

Performing the partial differentiation of (28b) and (28c) with respect to θ gives

$$\frac{\partial\rho^2}{\partial\theta} = -2a^2\sin\theta\cos\theta \tag{36a}$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{\Sigma^2}\right) = \frac{2a^2\Delta}{\Sigma^4}\sin\theta\cos\theta. \tag{36b}$$

Substituting (36*a*) and (36*b*) into (35), and simplifying a bit, leads to

$$g_{00,2} = -\frac{2a^2\Delta}{\Sigma^4}\left(\Sigma^2 - \rho^2\Delta\right)\sin\theta\cos\theta. \tag{37}$$

Then, substituting (37) into (30), and using $g^{22} = -1/\rho^2$ and $\mathbf{g}_2 = \mathbf{e}_\theta\rho$, puts the $\theta-$component of the force in the form

$$\mathbf{F}_\theta = -Gm\mathbf{e}_\theta\frac{2Mra^2}{\rho^3\Sigma^2}\left(r^2 + a^2\right)\sin\theta\cos\theta \tag{38}$$

where again we have put $M \to GM/c^2$ and used $E = mc^2$.

We are now in a position to state the entire expression of the force acting on the observer. Upon combining (34) and (38), the total force on the observer assumes the form

$$\begin{aligned}\mathbf{F} = -Gm\mathbf{e}_r&\left[\frac{M\rho^2\left(r^4 - a^4\right) + 2a^2r^2M\Delta\sin^2\theta}{\rho^3\Sigma^2\sqrt{\Delta}}\right] + \ldots\\ &- Gm\mathbf{e}_\theta\frac{2Mra^2}{\rho^3\Sigma^2}\left(r^2 + a^2\right)\sin\theta\cos\theta.\end{aligned} \tag{39}$$

Equation (39), derived on the basis of (12), is in agreement with the literature [10]. As a final consideration, it is a worthwhile exercise to verify that (39) reduces to the expected Schwarzschild solution when the angular momentum, a, is set equal to zero. Putting $a \to 0$ in (28*a*), (28*b*), and (28*c*) respectively gives $\Delta \to r^2 - 2GM/c^2r^2$, $\rho^2 \to r^2$, and $\Sigma^2 \to r^4$. Using these limiting expressions in (39) and simplifying a bit leads directly to

$$\mathbf{F} = -\frac{GMm}{r^2}\left(1 - \frac{2GM}{c^2r}\right)^{-\frac{1}{2}}\mathbf{e}_r. \tag{40}$$

Equation (40) is identical to (25), derived directly in Schwarzschild coordinates. Thus, in the limit $a \to 0$, (39) reduces to the expected expression of the force on an observer held stationary in Schwarzschild coordinates.

5. Rotating coordinates

The previous sections have demonstrated the use of (12) and (16) in various coordinate systems; however, it is also instructive to see how the force can be derived by way of (10). Let us use (10) to determine the force on an observer moving with a uniform velocity in a coordinate system which rotates with a constant angular velocity $\vec{\omega} = \omega\mathbf{e}_z$.

The metric tensor and inverse metric tensor in rotating coordinates are respectively given by [18, 19]

$$g_{\mu\nu} = \begin{pmatrix} 1-(\omega r/c)^2 & \omega y/c & -\omega x/c & 0\\ \omega y/c & -1 & 0 & 0\\ -\omega x/c & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{41a}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & \omega y/c & -\omega x/c & 0 \\ \omega y/c & -1+(\omega y/c)^2 & -xy(\omega/c)^2 & 0 \\ -\omega x/c & -yx(\omega/c)^2 & -1+(\omega x/c)^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{41b}$$

where $r^2 = x^2 + y^2$, and r is the radial distance from the origin of the rotating system to the observer. Expressing (3) in terms of the components of (41a) and (41b) gives the non-zero Christoffel symbols in the rotating system as

$$\begin{aligned} &\Gamma^1_{00} = -x(\omega/c)^2, \quad \Gamma^2_{00} = -y(\omega/c)^2, \\ &\Gamma^1_{20} = -\omega/c, \qquad\quad \Gamma^1_{02} = -\omega/c, \\ &\Gamma^2_{10} = \omega/c, \qquad\qquad \Gamma^2_{01} = \omega/c. \end{aligned} \tag{42}$$

Expressing (10) as a vector, in absence of external forces, and retaining only non-zero terms gives

$$\begin{aligned} \mathbf{F} = &-\mathbf{e}_1\left(\Gamma^1_{00}\int T^{00}d^3x + 2\Gamma^1_{20}\int T^{20}d^3x\right) + \ldots \\ &-\mathbf{e}_2\left(\Gamma^2_{00}\int T^{00}d^3x + 2\Gamma^2_{10}\int T^{10}d^3x\right) \end{aligned} \tag{43}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit basis vectors respectively pointing in the $x-$ and $y-$coordinate directions. Substituting (42) into (43), and rearranging terms, leads to

$$\mathbf{F} = \left(\frac{\omega}{c}\right)^2 r\mathbf{e}_r \int T^{00}d^3x + 2\frac{\omega}{c}\left(\mathbf{e}_1\int T^{20}d^3x - \mathbf{e}_2\int T^{10}d^3x\right). \tag{44}$$

Equation (44) can be simplified upon noticing that the integral in the first term on the right-hand side is just the total energy associated with the observer, whereas the second and third terms are respective components of the observer's momentum in the $y-$ and $x-$coordinate directions. Using this knowledge and also noticing that $\vec{\omega} \times (\vec{\omega} \times \mathbf{r}) = -\omega^2 r\mathbf{e}_r$, the force on the observer can be put in the form [15]

$$\mathbf{F} = -\frac{m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}) + 2m\,(\vec{\omega} \times \mathbf{v})}{(1 - v'^2/c^2)} \tag{45}$$

where $\mathbf{v}$ is the velocity of the observer in the rotating system, $\mathbf{v}' = \mathbf{v} + \vec{\omega} \times \mathbf{r}$ is the total velocity of observer relative to non-rotating, Minkowski space-time, and m is the total mass of the observer in the rotating system. Upon specializing to the case of non-relativistic velocities, $v' << c$, (45) simplifies to

$$\mathbf{F} \approx -m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}) - 2m\,(\vec{\omega} \times \mathbf{v}). \tag{46}$$

The first term on the right-hand side of (46) clearly is the centrifugal force on the observer, and the second term is the Coriolis force. As expected, (10) produces results consistent with known theory.

6. Discussion

As mentioned in the introduction, many textbooks dealing with general relativity do not emphasize the derivation of forces in enough detail. The analyses presented in the previous sections demonstrate straightforward methods for computing forces by way of general relativity. In section 2, an observer residing in arbitrary coordinates was used to demonstrate the relationship between forces and space-time geometry [1]-[9]. Covariant divergence of the stress-energy-momentum tensor was used to derive a general expression of the force experienced by the observer. An alternative form of the general force was then derived by using the lapse function to relate the quantity of proper time that elapses per unit of coordinate time [10].

In section 3, the general force and lapse function were used in familiar coordinate systems. The section began by considering the local co-moving coordinate system of a uniformly accelerating observer. Using the general force and the components of the metric tensor in the accelerating system led to an expression of the inertial force experienced by the accelerating observer. Specializing to the case of weak, Newtonian acceleration produced an expression of the force equivalent to that obtainable by using Newton's second law of motion. Next, an observer held stationary near a large gravitational source was considered. The lapse function was expressed in Schwarzschild coordinates and then used with the general force of section 2, leading to the well known general relativistic expression of the gravitational force [10]. The familiar Newtonian expression of the force was retrieved upon specializing to the case of a weakly gravitating source.

Section 4 was devoted to deriving the force on an observer near a rotating, Kerr black hole. The general force was applied in Boyer-Lindquist coordinates for the case of an observer being dragged around the black hole. This led to an expression of the force in agreement with the literature [10]. It was then shown that when the angular momentum of the black hole is put equal to zero, the force on the observer reduces to the expression of the force on an observer held stationary in Schwarzschild coordinates.

As a final consideration, in section 5 a form of the general force expressed in terms of the stress-energy-momentum tensor and Christoffel symbols was used to derive the force on an observer moving in rotating coordinates. Expressing the Christoffel symbols in rotating coordinates led to an expression of the force in terms of the total energy and momentum of the observer in the rotating system. Specializing to the case of slowly rotating coordinates led to well known expressions of the centrifugal and Coriolis forces on the observer.

Appendix

When working in general coordinates, it is helpful to have the gradient operator already worked out ahead of time. The gradient of an arbitrary scalar function, ψ, is expressed in general coordinates as [15]

$$\nabla\psi = -\mathbf{g}_i g^{ij} \partial_j \psi \tag{A.1}$$

where g^{ij} is the inverse metric tensor, ∂_j denotes partial differentiation with respect to a general coordinate q^j, $\mathbf{g}_i$ is a general basis vector pointing in the $i-$coordinate direction, and Latin indices are taken over the values $(1, 2, 3)$. Summing over the indices i and j puts (A.1) in the expanded form

$$
\begin{aligned}
-\nabla\psi = \mathbf{g}_1 \left(g^{11}\partial_1\psi + g^{12}\partial_2\psi + g^{13}\partial_3\psi \right) + \ldots \\
\mathbf{g}_2 \left(g^{21}\partial_1\psi + g^{22}\partial_2\psi + g^{23}\partial_3\psi \right) + \ldots \\
\mathbf{g}_3 \left(g^{31}\partial_1\psi + g^{32}\partial_2\psi + g^{33}\partial_3\psi \right)
\end{aligned}
\tag{A.2}
$$

where the minus sign has been moved to the left-hand side of (A.2) for the sake of simplicity. For the special case when the metric tensor g_{ij} is diagonal, (A.2) simplifes greatly, leaving us with

$$
-\nabla\psi = \mathbf{g}_1 g^{11}\partial_1\psi + \mathbf{g}_2 g^{22}\partial_2\psi + \mathbf{g}_3 g^{33}\partial_3\psi. \tag{A.3}
$$

In addition, when g_{ij} is diagonal we can put

$$
g^{ii} = \frac{1}{g_{ii}} = \frac{1}{\sqrt{-g_{ii}}}\frac{1}{\sqrt{-g_{ii}}}. \tag{A.4}
$$

Using (A.4) in (A.3) and putting $\mathbf{e}_i = \mathbf{g}_i/\sqrt{g_{ii}}$ leads to

$$
-\nabla\psi = \frac{\mathbf{e}_1}{\sqrt{-g_{11}}}\partial_1\psi + \frac{\mathbf{e}_2}{\sqrt{-g_{22}}}\partial_2\psi + \frac{\mathbf{e}_3}{\sqrt{-g_{33}}}\partial_3\psi \tag{A.5}
$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are basis vectors of unit length pointing in their respective coordinate directions. Upon introducing the Lamé coefficients, $h_i = \sqrt{-g_{ii}}$, (A.5) assumes the familiar form of the gradient in orthogonal coordinates:

$$
-\nabla\psi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\psi}{\partial q^1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\psi}{\partial q^2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\psi}{\partial q^3}. \tag{A.6}
$$

Thus, either (A.5) or (A.6) can be used when the metric tensor g_{ij} is diagonal, but when the coordinate system is non-orthogonal, one must resort to using the full expression of the gradient given by (A.2).

With suitable expressions of the gradient operator in hand, let us derive the gradient operator in Schwarzschild coordinates as a simple example. In Schwarzschild coordinates, the metric tensor is given by

$$
g_{\mu\nu} = \begin{pmatrix}
1 + \frac{2\phi}{c^2} & 0 & 0 & 0 \\
0 & -\left(1 + \frac{2\phi}{c^2}\right)^{-1} & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
0 & 0 & 0 & -r^2\sin^2\theta
\end{pmatrix} \tag{A.7}
$$

where ϕ is the gravitational field potential outside a large, spherically symmetric body of mass M. The inverse metric tensor in Schwarzschild coordinates is

$$
g^{\mu\nu} = \begin{pmatrix}
\left(1 + \frac{2\phi}{c^2}\right)^{-1} & 0 & 0 & 0 \\
0 & -\left(1 + \frac{2\phi}{c^2}\right) & 0 & 0 \\
0 & 0 & -r^{-2} & 0 \\
0 & 0 & 0 & -r^{-2}\sin^{-2}\theta
\end{pmatrix}. \tag{A.8}
$$

Since the metric tensor $g_{\mu\nu}$ is diagonal, the gradient operator may be derived by way of (A.5). Upon substituting the components of (A.7) directly into (A.5), it is straightforward to see that the gradient operator assumes the form

$$\nabla \to \left(1 + \frac{2\phi}{c^2}\right)^{1/2} \mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\mathbf{e}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \tag{A.9}$$

where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ are respective vectors of unit length in the (r, θ, φ) directions.

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