Archimedes' Principle in General Coordinates

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Abstract. Archimedes' principle is well known to state that a body submerged in a fluid is buoyed up by a force equal to the weight of the fluid displaced by the body. Herein, Archimedes' principle is derived from first principles by using conservation of the stress-energy-momentum tensor in general coordinates. The resulting expression for the force is applied in Schwarzschild coordinates and in rotating coordinates. Using Schwarzschild coordinates for the case of a spherical mass suspended within a perfect fluid leads to the familiar expression of Archimedes' principle. Using rotating coordinates produces an expression for a centrifugal buoyancy force that agrees with accepted theory. It is then argued that Archimedes' principle ought to be applicable to non-gravitational phenomena, as well. Conservation of the energy-momentum tensor is then applied to electromagnetic phenomena. It is shown that a charged body submerged in a charged medium experiences a buoyancy force in accordance with an electromagnetic analog of Archimedes' Principle.

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1. Introduction

As is well known, Archimedes' principle states that a body submerged in a fluid is buoyed up by a force equal to the weight of the fluid displaced by the body. When such a body is suspended in the fluid, the apparent weight of the body is given by the familiar expression

$$\mathbf{F} = m\mathbf{g}\left(1 - \frac{\rho_f}{\rho}\right) \tag{1}$$

where m and ρ are respectively the mass and mass-density of the body, $\mathbf{g}$ is the gravitational field vector, and ρ_f is the mass-density of the displaced fluid. A preliminary search of the literature reveals that very rarely, if at all, has (1) been derived from first principles without first stating Archimedes' principle [1, 3], and certainly not by employing the machinery of general relativity. Archimedes' principle, embodied by (1), is a statement about a particular gravitational phenomenon; and as such, ought to be derivable on the basis of known gravitational theory without having to rely on a foreknowledge of Archimedes' principle. It is envisioned that the analysis presented in the following sections will prove useful in teaching both Archimedes' principle and general relativity, as well as providing students with an example of how general relativity can be employed in the analysis of familiar problems encountered in the laboratory.

In the next section, conservation of energy-momentum is used in general coordinates to derive an equation of motion of a spherical mass suspended within a perfect fluid. Carrying this out in general coordinates produces a rather long equation. It is shown that the equation of motion reduces to a very simple form so long as the spherical mass and the fluid remain stationary, and the metric tensor is time independent. Rather than drawing heavily on equations from hydrostatics texts, as is often done [2], the equation of motion is taken aside and recast for the case when the sphere is absent. Doing this leads to an equation of motion solely for the fluid. Returning to the equation of motion of the sphere and taking the fluid to be in static equilibrium leads to an expression for the force on the sphere in general coordinates.

Section 3 is devoted to demonstrating the use of the expression for the force derived in the previous section. The section begins with a consideration of Schwarzschild coordinates. Using the metric tensor in Schwarzschild coordinates and limiting things to first order in the gravitational potential leads to an expression for the force identical to (1). Next, the rotating coordinate system is considered. The sphere and fluid are taken to be at rest relative to one another, and thus rotate together with the coordinate system. Using the general expression of the force with the metric tensor in rotating coordinates leads to an interesting expression of the force referred to in the literature as a centrifugal buoyancy force [4, 5]. As a final consideration, the expression of the force derived in the previous section is applied to the case of two gravitating bodies in a uniformly distributed material medium. It is shown that each body is buoyed by the medium due to the gravitational field of the other body.

In Sec. 4, an electromagnetic analog of Archimedes' principle is presented. An equation of motion of a sphere is derived by considering the difference of the electromagnetic energy-momentum tensor inside and outside the spherical boundary surface of the sphere. Using this approach leads quite simply to an equation of motion expressed in terms of the familiar electromagnetic field tensor. The force on the sphere is then expressed in vector notation, which demonstrates a form strikingly similar to (1). As a final thought, it is shown that when the total charge within the sphere is zero, the sphere is buoyed up by an electrostatic force, in direct analogy with Archimedes' principle.

2. Buoyancy in General Coordinates

As pointed out in the Introduction, conservation of the stress-energy-momentum tensor can be used to derive the force on a body suspended within a fluid without having to consider any preconditions. When working in the presence of gravitation, the conservation of energy-momentum is expressed in covariant form as

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{2}$$

where $T^{\mu\nu}$ is the stress-energy-momentum tensor, the semi-colon (;) denotes covariant differentiation, and Greek indices are carried over the set of values $(0, 1, 2, 3)$. Expanding the covariant derivative gives

$$\partial_\nu T^{\mu\nu} + \Gamma^{\mu}_{\alpha\nu} T^{\alpha\nu} + \Gamma^{\nu}_{\alpha\nu} T^{\mu\alpha} = 0 \tag{3}$$

in which ∂_ν denotes partial differentiation with respect to x^ν, and the Christoffel symbol $\Gamma^{\alpha}_{\mu\nu}$ is defined in terms of the metric tensor as

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\beta,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta} \right). \tag{4}$$

Writing the Christoffel symbols in terms of the metric tensor puts (3) in the form

$$\begin{aligned} \partial_\nu T^{\mu\nu} &+ \frac{1}{2} g^{\mu\gamma} \left(g_{\gamma\alpha,\nu} + g_{\nu\gamma,\alpha} - g_{\alpha\nu,\gamma} \right) T^{\alpha\nu} + \ldots \\ &\quad + \frac{1}{2} g^{\nu\gamma} \left(g_{\gamma\alpha,\nu} + g_{\nu\gamma,\alpha} - g_{\alpha\nu,\gamma} \right) T^{\mu\alpha} = 0. \end{aligned} \tag{5}$$

Combining symmetric terms and rearranging a bit then leads to

$$\partial_\nu T^{\mu\nu} + g^{\mu\gamma} \left(g_{\gamma\alpha,\nu} - \frac{1}{2} g_{\alpha\nu,\gamma} \right) T^{\alpha\nu} + \frac{1}{2} g^{\nu\gamma} g_{\nu\gamma,\alpha} T^{\mu\alpha} = 0. \tag{6}$$

Now let us consider a spherical body of mass, m, and density, ρ, suspended within a perfect fluid of density, ρ_f. It is envisioned that a thin tether supports the sphere within the fluid, and that the fluid and sphere are allowed to reach thermodynamic equilibrium. In applying (6), it is important to note that the stress-energy-momentum tensor is conserved for the complete system, not just for the mass or the fluid alone. This suggests that the stress-energy-momentum tensor for the mass-fluid system should be expressed as [6]

$$T^{\mu\nu} = T^{\mu\nu}_{m} + T^{\mu\nu}_{f} \tag{7}$$

where $T^{\mu\nu}_m$ and $T^{\mu\nu}_f$ are the stress-energy-momentum tensors of the mass and the perfect fluid, respectively. At this point, it should be mentioned that $T^{\mu\nu}_m$ and $T^{\mu\nu}_f$ include all forms of energy and momentum, as well as any internal stress, associated with the sphere and the fluid, respectively. Moreover, $T^{\mu\nu}_m$ should be understood to include the effects of any external forces acting on the sphere, such as due to the tether holding the sphere stationary relative to the fluid.

Substituting (7) into (6) and then summing over all the indices in (6) leads to a very long expression. A shortcut, however, is to recognize that for the special case of stationary bodies, we may put $u^\mu = (c, 0, 0, 0)$; and when the metric tensor is independent of time, we have $g_{\mu\nu,0} = 0$. Carrying out this shortcut method leaves (6) in the considerably simplified form

$$\partial_\nu T^{\mu\nu} - \frac{1}{2} g^{\mu j} g_{00,j} T^{00} = 0 \tag{8}$$

wherein Latin indices are taken over the values $(1, 2, 3)$. It is worthwhile to point out that although $u^\mu = (c, 0, 0, 0)$ may be a constant, there is no guarantee that $\partial_\nu u^\mu = 0$ [7]. For this reason, the first term in (8) is left unchanged by the shortcut described above. Summing over ν in the left-most term in (8) and then integrating over the volume of the sphere gives

$$\frac{1}{c}\frac{d}{d\tau}\int T^{\mu 0} d^3x + \int \partial_j T^{\mu j} d^3x - \frac{1}{2} g^{\mu j} g_{00,j} \int T^{00} d^3x = 0 \tag{9}$$

in which τ is proper time. Additionally, the metric tensor has been moved outside the integral in the right-most term on the assumption that the gravitational potential is roughly uniform across the volume of the sphere [6].

With (9) in hand, it is straightforward to see that for $T^{\mu\nu}_m$, the second integral must be zero since the sphere doesn't convey energy or momentum across the boundary surface of the sphere. For $T^{\mu\nu}_f$, the first and third integrals are zero because there is no fluid within the volume of the sphere. Following this line of reasoning, (9) can be recast in the form

$$\frac{1}{c}\frac{d}{d\tau}\int T^{\mu 0}_m d^3x + \int \partial_j T^{\mu j}_f d^3x - \frac{1}{2} g^{\mu j} g_{00,j} \int T^{00}_m d^3x = 0. \tag{10}$$

Upon expressing the four-momentum of the sphere as

$$P^\mu = \frac{1}{c}\int T^{\mu 0}_m d^3x \tag{11}$$

the first term in (10) is easily identified as the four-force on the sphere due to the tether holding the sphere stationary in the fluid. Using (11) and $F^\mu = dP^\mu/d\tau$, puts (10) in the form

$$F^\mu + \int \partial_j T^{\mu j}_f d^3x - \frac{1}{2} g^{\mu j} g_{00,j} \int T^{00}_m d^3x = 0. \tag{12}$$

In order to simplify the second term in (12), let us return to (9) and consider the case in which the sphere is absent. Using (9) for the fluid, and dropping the volume integration, gives

$$\frac{1}{c}\frac{dT^{\mu 0}_f}{d\tau} + \partial_j T^{\mu j}_f - \frac{1}{2} g^{\mu j} g_{00,j} T^{00}_f = 0. \tag{13}$$

This is the equation of motion of the fluid in absence of the sphere. When the fluid is in static equilibrium, we have $dT_f^{\mu 0}/d\tau = 0$ in (13), which then gives

$$\partial_j T_f^{\mu j} = \frac{1}{2} g^{\mu j} g_{00,j} T_f^{00}. \tag{14}$$

Substituting (14) into the second term in (12) and then solving for the vector force on the sphere leads directly to

$$\mathbf{F} = \frac{mc^2}{2} \frac{g^{ij} g_{00,j}}{g_{00}} \left(1 - \frac{\rho_f}{\rho}\right) \mathbf{g}_i \tag{15}$$

in which $T^{00} = \rho(U^0)^2$ and $g_{\mu\nu}U^\mu U^\nu = c^2$ have been used, and $\mathbf{g}_i$ is a basis vector pointing in the $i-$coordinate direction.

Equation (15) is the apparent weight of the sphere one would measure upon suspending the sphere within the fluid. Moreover, (15) is applicable in any coordinate system so long as the fluid and the sphere remain stationary within the system, and the metric tensor is time-independent. Equation (1) is merely a special case of (15), arising when an appropriate coordinate system is used for the metric tensor. Perhaps, the best way to see the relationship between (1) and (15) is to go ahead and apply (15) to familiar examples. In the next section, (15) is demonstrated in Schwarzschild coordinates and then again in rotating coordinates. In both cases (15) gives results consistent with accepted theory.

3. Buoyancy in Schwarzschild and Rotating Coordinate Systems

We wish to express (15) in Schwarzschild coordinates to obtain an expression of Archimedes' principle consistent with (1). The metric tensor and the inverse metric tensor in Schwarzschild coordinates can be expressed in the form [8]

$$g_{\mu\nu} = \begin{pmatrix} 1 + \frac{2\phi}{c^2} & 0 & 0 & 0 \\ 0 & -\left(1 + \frac{2\phi}{c^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \tag{16a}$$

$$g^{\mu\nu} = \begin{pmatrix} \left(1 + \frac{2\phi}{c^2}\right)^{-1} & 0 & 0 & 0 \\ 0 & -\left(1 + \frac{2\phi}{c^2}\right) & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2} \sin^{-2}\theta \end{pmatrix} \tag{16b}$$

where ϕ is the field potential of the gravitational source. Summing over j in (15) gives

$$\mathbf{F} = \frac{mc^2}{2g_{00}} \left(1 - \frac{\rho_f}{\rho}\right) \left(g^{i1} g_{00,1} + g^{i2} g_{00,2} + g^{i3} g_{00,3}\right) \mathbf{g}_i. \tag{17}$$

Upon using the components of (16a) and (16b) in (17), it is straightforward to see that only the radial, $i = 1$ component of the force is non-zero. Carrying this out and noting that the gradient operator in Schwarzschild coordinates is [9]

$$\nabla \to \left(1 + \frac{2\phi}{c^2}\right)^{1/2} \mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\mathbf{e}_\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \tag{18}$$

puts (17) in the form

$$\mathbf{F} = -m \left(1 - \frac{\rho_f}{\rho}\right) \left(1 + \frac{2\phi}{c^2}\right)^{-1} \nabla\phi. \tag{19}$$

Limiting (19) to first order in ϕ and putting $\mathbf{g} = -\nabla\phi$ leads to an expression identical to (1):

$$\mathbf{F} = m\mathbf{g} \left(1 - \frac{\rho_f}{\rho}\right). \tag{20}$$

Thus, (1) and (20) are a particular solution arising when (15) is expressed in Schwarzschild coordinates. This suggests that (15) is a more general statement of Archimedes' principle than (1).

Another familiar coordinate system is one that rotates with a constant angular velocity, ω. Imagine that the fluid is rotating, such as in a centrifuge, and the sphere is suspended from the origin of the rotating system into the fluid. The fluid and the sphere remain stationary relative to one another, and thus are rotating together with the coordinate system. Such a configuration is well known to give rise to a buoyancy force on the sphere, sometimes referred to as a centrifugal buoyancy force [4, 5]. In rotating coordinates, the metric tensor and the inverse metric tensor are [10]

$$g_{\mu\nu} = \begin{pmatrix} 1 - v^2 & -v_x & -v_y & 0 \\ -v_x & -1 & 0 & 0 \\ -v_y & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{21a}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & -v_x & -v_y & 0 \\ -v_x & -1 + v_x^2 & v_x v_y & 0 \\ -v_y & v_x v_y & -1 + v_y^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{21b}$$

wherein $v_x = -\omega y$, $v_y = \omega x$, $v^2 = v_x^2 + v_y^2$, and we have put $c = 1$ for the sake of simplicity. Summing over i in (17) and retaining only non-zero terms gives the force as

$$\mathbf{F} = \frac{m}{2g_{00}} \left(1 - \frac{\rho_f}{\rho}\right) \left[\left(g^{11} g_{00,1} + g^{12} g_{00,2}\right) \mathbf{g}_1 + \left(g^{21} g_{00,1} + g^{22} g_{00,2}\right) \mathbf{g}_2\right]. \tag{22}$$

Substituting the components given in (21a) and (21b) into (22), and then performing some algebraic manipulation leads to

$$\mathbf{F} = \frac{m\omega^2}{(1 - v^2)} \left(1 - \frac{\rho_f}{\rho}\right) (x\mathbf{e}_x + y\mathbf{e}_y). \tag{23}$$

wherein $\mathbf{e}_x$ and $\mathbf{e}_y$ are respective unit vectors in the $x-$ and $y-$coordinate directions. Thus, it is straightforward to see that the first order vector-force on the sphere in the rotating coordinate system is

$$\mathbf{F} = m\omega^2 r \left(1 - \frac{\rho_f}{\rho}\right) \mathbf{e}_r \tag{24}$$

in which r is the radial distance from the origin of the rotating system to the sphere, and $\mathbf{e}_r$ is a unit vector in the $r-$direction. Equation (24) is the well known expression of the centrifugal buoyancy force for a rotating fluid [4, 5]. As a final thought, upon putting $\vec{\omega} = \omega \mathbf{e}_z$ and noting that $\vec{\omega} \times (\vec{\omega} \times \mathbf{r}) = -\omega^2 r \mathbf{e}_r$, it is straightforward to see that (24) can be expressed equivalently as

$$\mathbf{F} = -m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}) \left(1 - \frac{\rho_f}{\rho}\right). \tag{25}$$

Equation (25) clearly illustrates the centrifugal nature of the buoyancy force on the sphere in rotating coordinates.

Another interesting application of (15) is for the case when two bodies of comparable size reside in a uniformly distributed material medium. Let us consider two spherical bodies, called A and B, having respective mass-densities ρ_A and ρ_B, residing within a uniformly distributed material medium, having a mass-density ρ_f. The centers of the bodies are separated by a distance r. So long as the bodies are held stationary, and their respective gravitational fields are time-independent, the vector force on either body due to the gravitational field of the other body and the medium can be expressed in terms of (15). Applying (15) to body A, and limiting things to first order terms, puts the force on body A in the form

$$\mathbf{F}_A = -\frac{m_A c^2}{2} \delta^{ij} g_{00,j} \left(1 - \frac{\rho_f}{\rho_A}\right) \mathbf{e}_i \tag{26}$$

in which $\delta^{ij} = 1$ when $i = j$, and $\delta^{ij} = 0$ when $i \neq j$. The metric tensor $g_{\mu\nu}$ expresses the space-time curvature due to the combined influence of body B and the intervening medium. Were the two bodies situated in a vacuum, $g_{\mu\nu}$ would be expressed in Schwarzschild coordinates. With the medium present, however, we should expect an additional contribution to $g_{\mu\nu}$.

According to N. K. Kofinti [11], g_{00} outside an empty spherical cavity in a uniformly distributed material medium is given by

$$g_{00} = 1 + \frac{2Gm_f}{rc^2} \tag{27}$$

where m_f is the mass of the portion of the medium that would otherwise fill the volume of the cavity. With body B occupying the cavity, the superposition principle suggests that we may put

$$g_{00} = 1 - \frac{2G}{rc^2}(m_B - m_f). \tag{28}$$

It is worthwhile to note that (28) reduces to the familiar Schwarzschild solution when $m_B \neq 0$ and $m_f \to 0$; whereas when $m_B \to 0$ and $m_f \neq 0$, (28) reduces to (27). Since

(28) is strictly a function of r, it is easy to see that only the $r-$component of (26) is non-zero. Carrying out the differentiation of (28) with respect to r, and substituting into (26), leads to [12, 13]

$$\mathbf{F} = -G\frac{m_A m_B}{r^2}\left(1 - \frac{\rho_f}{\rho_A}\right)\left(1 - \frac{\rho_f}{\rho_B}\right)\mathbf{e}_r \tag{29}$$

in which the subscript A has been dropped from $\mathbf{F}_A$ in recognition that the force is mutual between bodies A and B.

Equation (29) makes some interesting predictions. As expected, when both ρ_A and ρ_B are greater than ρ_f, the force is attractive. Interestingly, however, when m_A and m_B are both zero, the force is still attractive. Thus, if the two bodies are voids in the medium, they attract one another. Furthermore, when either body has a density greater than ρ_f and the other body has a density less than ρ_f, the force between the two bodies is repulsive. It is interesting that two voids attract one another while a void and a solid body will repel one another.

4. An Electromagnetic Analogy of Archimedes' Principle

A curious thought presents itself upon studying the derivation leading up to (15). As noted in the Introduction, Archimedes' principle states that the force on the submerged sphere is in essence, the difference between the weight of the sphere and the weight of the fluid displaced by the sphere. Under the right circumstances, shouldn't such a phenomenon occur with non-gravitational forces, as well? With regards to electromagnetism, one can envision a body having a different charge density than a surrounding charged medium. It seems to make sense that there ought to be a buoyancy force on the body, proportional to the charge displaced by the body. Let us take a closer look at Archimedes' principle as it applies to electromagnetism.

Consider an enclosed sphere characterized on the inside by an electromagnetic stress-energy-momentum tensor, $T^{\mu\nu}_{IN}$, residing within a region having an electromagnetic stress-energy-momentum tensor, $T^{\mu\nu}_{OUT}$. In the previous sections, the stress-energy due to the fluid was entirely outside the volume of the sphere. In the present case, however, there is electromagnetic stress inside as well as outside the sphere. Taking this into account, and neglecting any gravitational effects due to the energy-density of the bodies and fields, (2) becomes simply

$$\partial_\nu\left(T^{\mu\nu}_m + T^{\mu\nu}_{IN} - T^{\mu\nu}_{OUT}\right) = 0 \tag{30}$$

where $T^{\mu\nu}_m$ is the stress-energy-momentum tensor of the sphere, and the minus sign on the third term is included because $T^{\mu\nu}_{OUT}$ is an inward-directed stress on the boundary surface of the sphere. Equation (30) can be quickly simplified upon recalling that the equation of motion of electromagnetism can be written as [6, 14]

$$\partial_\nu T^{\mu\nu} = -F^\mu_\nu j^\nu \tag{31}$$

in which F^μ_ν is the mixed electromagnetic field tensor, given by [15]

$$F^\mu_\nu = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & B^3 & -B^2 \\ E^2/c & -B^3 & 0 & B^1 \\ E^3/c & B^2 & -B^1 & 0 \end{pmatrix} \tag{32}$$

and j^ν is the current-density four-vector, defined as $j^\nu = (c\rho,\ \mathbf{j})$. Integrating (30) over the volume of the sphere, and using (11) and (31) leads to

$$\frac{dP^\mu}{d\tau} - \int (F^\mu_\nu j^\nu_{IN} - F^\mu_\nu j^\nu_{OUT})\, d^3x = 0. \tag{33}$$

Assuming that the external fields are roughly uniform across the volume of the sphere, F^μ_ν can be moved outside of the integral [6] in the second term of (33). Carrying this out leads to an equation of motion of the form

$$\frac{dP^\mu}{d\tau} = F^\mu_\nu \int (j^\nu_{IN} - j^\nu_{OUT})\, d^3x. \tag{34}$$

We are now in a position to express the three-force on the sphere in terms of the external electric and magnetic fields, **E** and **B**. Summing over ν in (34) and using (32) leads to the $i-$component of the force, given by

$$F^i = \int (\rho_{IN}\mathbf{E} + \mathbf{j}_{IN} \times \mathbf{B})^i\, d^3x - \int (\rho_{OUT}\mathbf{E} + \mathbf{j}_{OUT} \times \mathbf{B})^i\, d^3x \tag{35}$$

in which $F^i = dP^i/d\tau$ has been used. Rearranging (35) so as to move the external fields, **E** and **B**, outside the integrals, and expressing the force in vector notation leads to

$$\mathbf{F} = \mathbf{E} \int (\rho_{IN} - \rho_{OUT})\, d^3x - \mathbf{B} \times \int (\mathbf{j}_{IN} - \mathbf{j}_{OUT})\, d^3x. \tag{36}$$

The right-most term in (36) is interesting, but goes beyond the scope of the present analysis. Rather, let us consider an idealistic, electrostatic case in which the sphere has a charge-density, ρ, the exterior medium has a charge-density, ρ_f, and both reside in an external, background electrostatic field, **E**. Putting $\mathbf{B} = 0$ in (36), and simplifying a bit, leads immediately to an expression of the force on the sphere, given by

$$\mathbf{F} = q_{IN}\mathbf{E}\left(1 - \frac{\rho_{OUT}}{\rho_{IN}}\right) \tag{37}$$

in which q_{IN} is the total charge contained within the volume of the sphere. Equation (37) clearly is an electrostatic analogy of (1). As a final thought, it is interesting to note that when $q_{IN} = 0$, (37) predicts that $\mathbf{F} = -q_{OUT}\mathbf{E}$, where q_{OUT} is the total charge displaced by the volume of the sphere. This suggests that a void within the charged medium experiences a buoyancy force, in direct analogy with Archimedes' principle.

In closing, it should be mentioned that the concept of applying Archimedes' principle to electromagnetic phenomena is not new. For example, dielectrophoresis (DEP) is an area of research in which external, time-varying electromagnetic fields are applied to dielectric compounds, so as to separate constituent substances having different values of the dielectric permittivity, ϵ. A quick search of the literature reveals that the force on a particle having a complex permittivity, ϵ_p, submerged within a medium having a complex permittivity, ϵ_m, experiences a force, given by [16, 17]

$$\mathbf{F}_{DEP} = 2\pi a^3 \epsilon_m Re\left(\frac{\epsilon_p - \epsilon_m}{\epsilon_p - 2\epsilon_m}\right)\nabla\left(\mathbf{E}^2_{RMS}\right) \tag{38}$$

in which a is the radius of the particle, and $\mathbf{E}_{RMS}$ is the root-mean square (RMS) value of the external electric field. Another similar area of research is electrophoresis, where the external field generally is static [18]. It is not too difficult to show that an expression similar to (38) holds for the case of a static electric field, with the exception that ϵ_m and ϵ_p are no longer complex, and the electric field, $\mathbf{E}$, is not time-averaged. Although investigating the relation between (36) and (38) would be interesting, such an analysis is left to the future.

5. Conclusions

As pointed out in the Introduction, Archimedes' principle has rarely been derived from first principles without first stating Archimedes' principle [1, 3], and certainly not by use of equations most often associated with general relativity. Herein, a more general approach to deriving Archimedes' principle has been demonstrated. Covariant conservation of energy-momentum for a spherical mass submerged in a perfect fluid was used to derive the buoyancy force on the sphere in general coordinates. Although the resulting expression for the force is expressed in terms of the metric tensor, and thus looks a bit different, it was pointed out that the usual expression for the buoyancy force is merely a special case derivable from the general force expression. This was demonstrated by expressing the general force in Schwarzschild coordinates, which led directly to the familiar expression for the buoyancy force on a submerged body. As a further example, the general force was then applied in rotating coordinates. Using the metric tensor in rotating coordinates with the general expression of the force led to an expression known in the literature as a centrifugal buoyancy force. In addition, the general force equation was applied to the case of two gravitating bodies of comparable size residing in a material medium. The resulting expression of the force suggests that each body experiences a buoyancy force equal to the weight of the displaced portion of the medium due to the gravitational field of the other body [12, 13].

As a final consideration, it was argued that conservation of energy-momentum ought to give rise to an analogous Archimedes' principle for non-gravitational phenomena, as well. Conservation of the stress-energy-momentum tensor was then applied in the case of electromagnetic phenomena. The resulting expression was used to show that a charged body submerged in a charged medium in the presence of an electrostatic field experiences a buoyancy force in accordance with an electromagnetic analog of Archimedes' principle.

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