Applying covariant versus contravariant electromagnetic tensors to rotating media

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When the covariant form of Maxwell's equations are applied to a rotating reference frame, a choice must be made to work with either a covariant electromagnetic tensor $F_{\alpha\beta}$ or a contravariant electromagnetic tensor $F^{\alpha\beta}$. We argue that which tensor one chooses is ultimately dictated by whether one chooses to express the electric and magnetic fields in terms of a vector basis or in terms of a one-form basis, dual to the vector basis. We explain that when fields are expressed as one-forms, the covariant electromagnetic tensor is used; and when fields are expressed as vectors, the contravariant tensor is used. Using this formalism, we derive general field equations expressed in terms of vector and one-form fields in the rotating and laboratory frames when matter is present. Fields in the presence of matter are then related to those in a vacuum by using a covariant form of Minkowski's constitutive equations, generalized to noninertial frames. Both vector and one-form field equations are used to derive the fields observed in the reference frame of a polarizable, permeable cylinder that rotates within an axially directed magnetic field. We find that the vector and one-form field equations both lead to predictions consistent with experimental results. We conclude that the choice between working with a covariant or contravariant electromagnetic tensor depends upon whether one chooses to express fields as vectors or as one-forms.

I. INTRODUCTION

In a recent paper,[1] we demonstrated one method by which relativistic electrodynamics can be extended to include rotating isotropic, linear media. Covariant field equations were used to derive general field equations in a rotating coordinate system. Fields in the presence of matter were then related to the electric and magnetic fields by use of a covariant form of Minkowski's constitutive equations.[2–5] We showed that using Minkowski's covariant constitutive equations in rotating coordinates leads directly to the correct constitutive equations and hence to the correct field equations in the rotating reference frame when matter is present.[6] We concluded that relativistic electrodynamics can be used for rotating linear media only when a covariant form of the constitutive equations is used.

In the course of performing that analysis, we chose to work with the covariant electromagnetic tensor $F_{\alpha\beta}$, composed of the components of the electric and magnetic fields. An alternative approach is to define the components of the electric and magnetic fields to comprise a contravariant electromagnetic tensor $F^{\alpha\beta}$. We pointed out that either the covariant or contravariant tensor can be used so long as one is consistent throughout the analysis. However, although both tensors lead to field equations of the same form in an inertial reference frame, when acceleration is present, such as in a rotating reference frame, the covariant and contravariant tensors lead to different sets of field equations.[7] Different predictions for the fields observed in a rotating reference frame are then obtained. This is confusing: all observers at rest in a particular reference frame ought to be able to agree on the form of the electric and magnetic fields observed in that frame. Thus, we expect that rotating observers working with a covariant electromagnetic tensor should obtain results that are consistent with rotating observers working with a contravariant electromagnetic tensor. It is an objective of this paper to provide a mathematical framework with which to explain why different field equations arise in a rotating frame, and to shed further light on the correct method by which relativistic electrodynamics can be applied to rotating media.

One important point to recognize is that fields, such as the electric and magnetic fields, can be expressed either in terms of a vector basis $\{\mathbf{e}_\alpha\}$ or in terms of a one-form basis $\{\tilde{\mathbf{w}}^\alpha\}$, dual to the $\{\mathbf{e}_\alpha\}$ according to $\tilde{\mathbf{w}}^\alpha(\mathbf{e}_\beta)=\delta^\alpha_\beta$.[8–10] As is very well known, any contravariant vector $\mathbf{v}$ can be expressed as a superposition of basis vectors $\{\mathbf{e}_\alpha\}$ as $\mathbf{v}=v^\alpha\mathbf{e}_\alpha$, where $\{\mathbf{e}_\alpha\}$ are distinct, linearly independent vectors. Similarly, any covariant vector $\tilde{\mathbf{v}}$ can be expressed as a superposition of basis one-forms $\{\tilde{\mathbf{w}}^\alpha\}$ as $\tilde{\mathbf{v}}=v_\alpha\tilde{\mathbf{w}}^\alpha$, where the basis one-forms $\{\tilde{\mathbf{w}}^\alpha\}$ are distinct and linearly independent. In general, the vector components v^α and the one-form components v_α are not equal, but instead are mapped into each other by use of the metric tensor. For example, vector components v^α can be mapped from the vector basis $\{\mathbf{e}_\alpha\}$ to a one-form basis $\{\tilde{\mathbf{w}}^\alpha\}$ by using the metric tensor: $v_\alpha=g_{\alpha\beta}v^\beta$. Or conversely, one-form components v_α can be mapped from the one-form basis $\{\tilde{\mathbf{w}}^\alpha\}$ to the vector basis $\{\mathbf{e}_\alpha\}$ by using the inverse metric tensor: $v^\alpha=g^{\alpha\beta}v_\beta$.

The same relationship holds true for second rank electromagnetic tensors $F^{\alpha\beta}$ and $F_{\alpha\beta}$. A basis for all second rank contravariant tensors is $\mathbf{e}_\alpha\otimes\mathbf{e}_\beta$, where $\otimes$ represents the outer product, and a basis for all second rank covariant tensors is $\tilde{\mathbf{w}}^\alpha\otimes\tilde{\mathbf{w}}^\beta$.[8] The components of contravariant tensors are vector components, and the components of covariant tensors are one-form components. Thus, whether one works with the covariant or contravariant electromagnetic tensor is actually dependent upon whether one chooses to express the electric and magnetic fields as vectors or as one-forms. When fields are expressed as vectors, the contravariant electromagnetic tensor $F^{\alpha\beta}$ is composed of the vector components of the electric and magnetic fields, $\{E^i,B^i\}$, and the covariant tensor is obtained by using the metric tensor to map those components onto the one-form basis according to $F_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} F^{\mu\nu}$. On the other hand, when fields are expressed as one-forms the covariant electromagnetic tensor $F_{\alpha\beta}$ is composed of the one-form components of the electric and magnetic fields, $\{E_i, B_i\}$, and the contravariant tensor is obtained by mapping the one-form components to the vector basis by using $F^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu}$. As pointed out above, the vector components and one-form components are not equal, and thus cannot be directly interchanged. In an inertial reference frame, this distinction is easily missed simply because Maxwell's field equations assume the same form with respect to either basis. However, in a noninertial frame wherein acceleration is present, expressing fields as vectors or as one-forms leads to different sets of field equations. It is our opinion that the confusion surrounding the choice of covariant versus contravariant electromagnetic tensors arises from a failure to fully recognize that the components of covariant and contravariant tensors are defined with respect to different bases.

In the next section we use the covariant form of Maxwell's equations in rotating coordinates to derive three-dimensional field equations for the rotating frame. We begin by considering the case when rotating observers express fields as one-forms. The covariant electromagnetic tensor is composed of the one-form components of the electric and magnetic fields, and the contravariant electromagnetic tensor is obtained by use of the inverse metric tensor in rotating coordinates. General field equations in three-dimensional notation are obtained, expressed in terms of one-form fields in the presence of matter. Next, we turn to the case in which rotating observers express fields as vectors. In this case, the contravariant electromagnetic tensor is composed of the vector components of the electric and magnetic fields, and the covariant tensor is obtained by using the metric tensor in rotating coordinates. General field equations are again obtained, but this time, expressed in terms of vector fields in the presence of matter.

In Sec. III we relate fields in the presence of matter to those in a vacuum by using a covariant form of the constitutive equations, first introduced by Minkowski in 1908.[2–5] We start by noting that, as with the field equations, one has a choice in expressing fields as one-forms or as vectors. We then point out that in order to maintain consistency throughout the analysis, we must use the same fields that were used in deriving the field equations. Adhering to this rule, we derive one-form and vector constitutive equations, and then substitute them into the corresponding set of field equations, derived in Sec. II. Carrying this out, we obtain one-form and vector field equations for the rotating frame in the presence of matter. We then bring Sec. III to a close by using a covariant form of the polarization and magnetization[1] to derive one-form and vector expressions for the polarization and magnetization in the rotating frame. We find that as with the constitutive equations, expressing fields as one-forms or as vectors leads to different forms for the polarization and magnetization in the rotating frame.

Section IV is devoted to deriving constitutive equations for a rotating material, observed in the laboratory reference frame. This is accomplished by using field transformations to transform the constitutive equations from the rotating frame to the lab frame. For both one-forms and vectors, we arrive at constitutive equations in agreement with Minkowski's constitutive equations, first obtained in 1908 by using special relativity for the case of uniform motion.[1,2,4,5,11] We then explain that observers in the lab frame can analyze experiments involving rotation by using either pair of constitutive equations in conjunction with Maxwell's field equations.

In Sec. V, we demonstrate the application of both field equations derived on the basis of one-forms and those derived on the basis vectors in a rotating frame. To do this, both sets of field equations are used to derive the fields observed in the reference frame of a hollow cylinder with electric permittivity ε and magnetic permeability μ that rotates within an external, axially directed magnetic field.[1,6,12] We assume that all fields are static, and that the cylinder is composed of a material that precludes free charges and currents. Carrying this out, we show that one-form fields and vector fields differ in form in the rotating frame, but assume the same form in the inertial frame of the laboratory.

In the Appendix, we derive field transformations that relate quantities in the rotating frame to those in the lab frame. We begin by noting that, although the rotating frame has a rotational velocity relative to the lab frame, at any given instant a momentarily comoving reference frame (MCRF) of an observer at rest in the rotating frame has a uniform velocity relative to the lab frame. Using this observation to our advantage, we first derive a relationship between fields in the rotating frame and those in the MCRF, and then use a Lorentz transformation to relate fields in the MCRF to those in the lab frame. Upon eliminating MCRF quantities between the two transformations, we obtain a direct transformation between the rotating and lab frames. We then transform the covariant and contravariant electromagnetic tensors from the lab frame to the rotating frame. Carrying this out, we arrive at transformations relating vector and one-form fields in the rotating frame to those in the lab frame.

II. FIELD EQUATIONS IN THE ROTATING REFERENCE FRAME

Although rotation does not lead to space–time curvature, a rotating reference frame is not a truly inertial frame of reference due to the presence of inertial forces.[6,13–16] Maxwell's equations can be extended to encompass accelerating frames by using the covariant field equations in arbitrary coordinates:[17,18]

$$\nabla_\alpha H^{\alpha\beta} = 4\pi j^\beta, \tag{1a}$$

$$\epsilon^{\mu\nu\kappa\lambda} \nabla_\nu F_{\kappa\lambda} = 0, \tag{1b}$$

where source charges and currents are given by the current four-vector j^β and we have used, and will continue to use, units in which the speed of light is set equal to unity: $c=1$. Noting that in rotating coordinates the covariant derivative is exactly equal to the ordinary partial derivative,[1] the field equations can be rewritten as

$$\partial_\alpha H^{\alpha\beta} = 4\pi j^\beta, \tag{2a}$$

$$\epsilon^{\mu\nu\kappa\lambda} \partial_\nu F_{\kappa\lambda} = 0. \tag{2b}$$

As can be seen, Eq. (2a) is expressed in terms of a contravariant electromagnetic tensor whereas Eq. (2b) is expressed in terms of a covariant electromagnetic tensor. Thus, one can choose to work with either the covariant electromagnetic tensor $F_{\alpha\beta}$ or the contravariant tensor $F^{\alpha\beta}$ so long as one is consistent throughout the analysis. As pointed out in the Introduction, which tensor one chooses is ultimately dictated by how one chooses to describe fields in the rotating reference frame. The electric and magnetic fields can be expressed in the usual way as vectors $\mathbf{E}=E^i\mathbf{e}_i$ and $\mathbf{B}=B^i\mathbf{e}_i$, or the fields can be expressed as one-forms $\tilde{\mathbf{E}}=E_i\tilde{\mathbf{w}}^i$ and $\tilde{\mathbf{B}}=B_i\tilde{\mathbf{w}}^i$.[8,9]

When the electric and magnetic fields are expressed as one-forms, the covariant field tensor $F_{\alpha\beta}$ assumes its familiar form as

$$F_{\alpha\beta}=\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{3}$$

and the contravariant tensor $F^{\alpha\beta}$ is obtained by raising indices with the inverse metric tensor:

$$F^{\alpha\beta}=g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}. \tag{4}$$

Using the inverse metric tensor in rotating coordinates, given in the Appendix, Eq. (4) leads to

$$F^{i0}=(\tilde{\mathbf{E}}-\mathbf{v}\times\tilde{\mathbf{B}}')_i, \tag{5a}$$

$$F^{ij}=-\epsilon^{ijk}(\tilde{\mathbf{B}}-\mathbf{v}\times(\tilde{\mathbf{E}}-\mathbf{v}\times\tilde{\mathbf{B}}'))_k, \tag{5b}$$

where quantities in the rotating frame carry a prime, and observers at rest in the rotating frame have velocity $\mathbf{v}=\omega r\mathbf{e}_\phi$ relative to the laboratory frame. Using Eqs. (3) and (5) in Eqs. (2), and limiting ourselves to the case of static fields, we find that the field equations are

$$\nabla\cdot(\tilde{\mathbf{D}}-\mathbf{v}\times\tilde{\mathbf{H}}')=4\pi\rho', \tag{6a}$$

$$\nabla\cdot\tilde{\mathbf{B}}'=0, \tag{6b}$$

$$\nabla\times\tilde{\mathbf{E}}'=0, \tag{6c}$$

$$\nabla\times(\tilde{\mathbf{H}}'-\mathbf{v}\times(\tilde{\mathbf{D}}'+\mathbf{v}\times\tilde{\mathbf{H}}'))=4\pi\mathbf{j}'. \tag{6d}$$

These are the general field equations for the rotating reference frame, written in three-dimensional notation with respect to a one-form basis $\{\tilde{\mathbf{w}}^\alpha\}$.

Conversely, when the electric and magnetic fields are expressed as vectors, the contravariant form of the field tensor $F^{\alpha\beta}$ is

$$F^{\alpha\beta}=\begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{7}$$

and the covariant tensor $F_{\alpha\beta}$ is then obtained by lowering indices with the metric tensor:

$$F_{\alpha\beta}=g_{\alpha\mu}g_{\beta\nu}F^{\mu\nu}. \tag{8}$$

Using the metric tensor in rotating coordinates gives

$$F_{0i}=(\mathbf{E}'+\mathbf{v}\times(\mathbf{B}'+\mathbf{v}\times\mathbf{E}'))^i, \tag{9a}$$

$$F_{ij}=\epsilon_{ijk}(\mathbf{B}'+\mathbf{v}\times\mathbf{E}')^k. \tag{9b}$$

This time, using Eqs. (7) and (9) in Eqs. (2), and again limiting ourselves to static fields, leads to

$$\nabla\cdot\mathbf{D}'=4\pi\rho', \tag{10a}$$

$$\nabla\cdot(\mathbf{B}'+\mathbf{v}\times\mathbf{E}')=0, \tag{10b}$$

$$\nabla\times(\mathbf{E}'+\mathbf{v}\times\mathbf{B}')=0, \tag{10c}$$

$$\nabla\times\mathbf{H}'=4\pi\mathbf{j}'. \tag{10d}$$

These are the three-dimensional field equations for the rotating reference frame, written with respect to a vector basis $\{\mathbf{e}_\alpha\}$.

III. CONSTITUTIVE EQUATIONS IN THE ROTATING FRAME

As yet, we have not specified a relationship between the auxiliary fields, **D** and **H**, and fundamental fields, **E** and **B**.[17] We can relate auxiliary fields in the presence of matter to the fundamental fields by using a covariant form of the constitutive equations, first introduced by Minkowski in 1908:[2–5]

$$H^{\lambda\mu}u_\mu=\epsilon F^{\lambda\mu}u_\mu, \tag{11a}$$

$$\epsilon^{\sigma\lambda\mu\nu}F_{\lambda\mu}u_\nu=\mu\epsilon^{\sigma\lambda\mu\nu}H_{\lambda\mu}u_\nu, \tag{11b}$$

where the electric permitivity ϵ and magnetic permeability μ are proper quantities defined in the local rest frame of the material. As in the case of the field equations, however, a choice must be made between working with a covariant or contravariant electromagnetic tensor. As pointed out in the two previous sections, this choice hinges on whether one chooses to express fields as one-forms or as vectors. Since the constitutive equations will be used in conjunction with the field equations, in order to maintain consistency throughout the analysis, we must use the same fields that were used in deriving the field equations.

When observers in the rotating frame express fields as one-forms, Eqs. (11) are used in conjunction with the covariant electromagnetic tensor given by Eq. (3). Generalizing to noninertial frames, Eqs. (11) are then

$$g^{\alpha\lambda}H_{\lambda\mu}u^\mu=\epsilon g^{\alpha\lambda}F_{\lambda\mu}u^\mu, \tag{12a}$$

$$\epsilon^{\sigma\lambda\mu\nu}F_{\lambda\mu}g_{\nu\delta}u^\delta=\mu\epsilon^{\sigma\lambda\mu\nu}H_{\lambda\mu}g_{\nu\delta}u^\delta. \tag{12b}$$

According to the metric tensor in rotating coordinates, given in the Appendix, an observer at rest in the rotating frame has a four-velocity $u^a=\gamma(1,0,0,0)$, where $\gamma=1/\sqrt{1-v^2}$. Using this four-velocity and the metric tensor in Eqs. (12), we find that auxiliary fields are related to fundamental fields according to

$$\tilde{\mathbf{D}}'=\epsilon\tilde{\mathbf{E}}', \tag{13a}$$

$$\tilde{\mathbf{H}}'=\frac{1}{\mu}\tilde{\mathbf{B}}'+\frac{1}{(1-v^2)}\left(\epsilon-\frac{1}{\mu}\right)\mathbf{v}\times\tilde{\mathbf{E}}', \tag{13b}$$

where quantities in the rotating frame carry a prime. Substituting Eqs. (13) into Eqs. (6) leads to

$$\nabla\cdot\left(\epsilon\tilde{\mathbf{E}}'-\mathbf{v}\times\frac{1}{\mu}\tilde{\mathbf{B}}'-\frac{1}{(1-v^2)}\left(\epsilon-\frac{1}{\mu}\right)\mathbf{v}\times(\mathbf{v}\times\tilde{\mathbf{E}}')\right)=4\pi\rho', \tag{14a}$$

$$\nabla\cdot\tilde{\mathbf{B}}'=0, \tag{14b}$$

$$\nabla\times\tilde{\mathbf{E}}'=0, \tag{14c}$$

$$\nabla\times\left(\frac{1}{\mu}(\tilde{\mathbf{B}}'-\mathbf{v}\times\tilde{\mathbf{E}}'+\mathbf{v}\times(\mathbf{v}\times\tilde{\mathbf{B}}'))\right)=4\pi\mathbf{j}'. \tag{14d}$$

These field equations are those used by observers at rest in the rotating frame that choose to express fields as one-forms.

Observers in the rotating frame choosing to express fields as vectors use Eqs. (11) in conjunction with the contravariant electromagnetic tensor given by Eq. (7). In this case, Eqs. (11) are written in the form

$$H^{\lambda\mu}g_{\mu\nu}u^{\nu}=\epsilon F^{\lambda\mu}g_{\mu\nu}u^{\nu}, \tag{15a}$$

$$\epsilon^{\sigma\lambda\mu\nu}g_{\lambda\alpha}g_{\mu\beta}F^{\alpha\beta}g_{\nu\delta}u^{\delta}=\mu\epsilon^{\sigma\lambda\mu\nu}g_{\lambda\alpha}g_{\mu\beta}H^{\alpha\beta}g_{\nu\delta}u^{\delta}. \tag{15b}$$

Carrying out the same procedure used to obtain Eqs. (13), we find that auxiliary and fundamental fields are related by

$$\mathbf{D}'=\epsilon\mathbf{E}'+\frac{1}{(1-\nu^2)}\left(\epsilon-\frac{1}{\mu}\right)\mathbf{v}\times\mathbf{B}', \tag{16a}$$

$$\mathbf{H}'=\frac{1}{\mu}\mathbf{B}'. \tag{16b}$$

Upon substituting these constitutive equations into Eqs. (10), we find that the field equations used by rotating observers that choose to express fields as vectors are

$$\nabla\cdot\left(\epsilon\mathbf{E}'+\frac{1}{(1-\nu^2)}\left(\epsilon-\frac{1}{\mu}\right)\mathbf{v}\times\mathbf{B}'\right)=4\pi\rho', \tag{17a}$$

$$\nabla\cdot(\mathbf{B}'+\mathbf{v}\times\mathbf{E}')=0, \tag{17b}$$

$$\nabla\times(\mathbf{E}'+\mathbf{v}\times\mathbf{B}')=0, \tag{17c}$$

$$\nabla\times\frac{1}{\mu}\mathbf{B}'=4\pi\mathbf{j}'. \tag{17d}$$

At this point, we should mention that Eqs. (11) can also be used to obtain expressions for the polarization **P** and magnetization **M** observed in the rotating reference frame.[1] Rotating observers that work with one-forms begin by taking

$$H_{\alpha\beta}=F_{\alpha\beta}-4\pi M_{\alpha\beta}, \tag{18}$$

where $M_{\alpha\beta}$ is the magnetization four-tensor composed of the components of the polarization and magnetization:[5]

$$M_{\alpha\beta}=\begin{pmatrix}0 & -P_1 & -P_2 & -P_3\\ P_1 & 0 & -M_3 & M_2\\ P_2 & M_3 & 0 & -M_1\\ P_3 & -M_2 & M_1 & 0\end{pmatrix}. \tag{19}$$

Substituting Eq. (18) into each of Eqs. (12) and solving for $M_{\alpha\beta}$ leads to

$$g^{\alpha\lambda}M_{\lambda\mu}u^{\mu}=\frac{1}{4\pi}(\epsilon-1)g^{\alpha\lambda}F_{\mu\lambda}u^{\mu}, \tag{20a}$$

$$\epsilon^{\sigma\lambda\mu\nu}M_{\lambda\mu}g_{\nu\delta}u^{\delta}=\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\epsilon^{\sigma\lambda\mu\nu}F_{\lambda\mu}g_{\nu\delta}u^{\delta}. \tag{20b}$$

Following the same steps used to derive the constitutive equations, Eqs. (13), we find that the polarization and magnetization in the rotating frame are

$$\tilde{\mathbf{P}}'=\frac{1}{4\pi}(\epsilon-1)\tilde{\mathbf{E}}', \tag{21a}$$

$$\tilde{\mathbf{M}}'=\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\tilde{\mathbf{B}}'+\frac{1}{4\pi(1-\nu^2)}\left(\frac{1}{\mu}-\epsilon\right)\mathbf{v}\times\tilde{\mathbf{E}}'. \tag{21b}$$

On the other hand, when fields are expressed as vectors, rotating observers use

$$H^{\alpha\beta}=F^{\alpha\beta}-4\pi M^{\alpha\beta}, \tag{22}$$

in which the contravariant magnetization four-tensor $M^{\alpha\beta}$ is defined as

$$M^{\alpha\beta}=\begin{pmatrix}0 & P^1 & P^2 & P^3\\ -P^1 & 0 & -M^3 & M^2\\ -P^2 & M^3 & 0 & -M^1\\ -P^3 & -M^2 & M^1 & 0\end{pmatrix}. \tag{23}$$

Substituting Eq. (22) into each of Eqs. (15) and then solving for $M^{\alpha\beta}$ gives

$$M^{\lambda\mu}g_{\mu\nu}u^{\nu}=\frac{1}{4\pi}(\epsilon-1)F^{\mu\lambda}g_{\mu\nu}u^{\nu}, \tag{24a}$$

$$\epsilon^{\sigma\lambda\mu\nu}g_{\lambda\alpha}g_{\mu\beta}M^{\alpha\beta}g_{\nu\delta}u^{\delta}=\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\epsilon^{\sigma\lambda\mu\nu}g_{\lambda\alpha}g_{\mu\beta}F^{\alpha\beta}g_{\nu\delta}u^{\delta}. \tag{24b}$$

The vector polarization and magnetization in the rotating frame are then

$$\mathbf{P}'=\frac{1}{4\pi}(\epsilon-1)\mathbf{E}'+\frac{1}{4\pi(1-\nu^2)}\left(\epsilon-\frac{1}{\mu}\right)\mathbf{v}\times\mathbf{B}', \tag{25a}$$

$$\mathbf{M}'=\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\mathbf{B}'. \tag{25b}$$

Upon comparing Eqs. (25) and (21), it is apparent that as with the constitutive equations, expressing fields as one-forms or as vectors leads to different forms for the polarization and magnetization in the rotating frame.

IV. CONSTITUTIVE EQUATIONS IN THE LABORATORY REFERENCE FRAME

We now turn to the problem of finding constitutive equations for a rotating medium, observed in the laboratory reference frame. To do this, we transform the constitutive equations from the rotating frame to the lab frame. As shown in the Appendix, one-form fields in the rotating frame are related to those in the lab frame according to

$$\tilde{\mathbf{E}}'=\tilde{\mathbf{E}}+\mathbf{v}\times\tilde{\mathbf{B}}-\frac{\gamma}{\gamma+1}(\mathbf{v}\cdot\tilde{\mathbf{E}})\mathbf{v}, \tag{26a}$$

$$\tilde{\mathbf{B}}'=\gamma(\tilde{\mathbf{B}}-\mathbf{v}\times\tilde{\mathbf{E}})+\gamma^2\mathbf{v}\times(\tilde{\mathbf{E}}+\mathbf{v}\times\tilde{\mathbf{B}})-\frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\tilde{\mathbf{B}})\mathbf{v}, \tag{26b}$$

$$\tilde{\mathbf{D}}'=\tilde{\mathbf{D}}+\mathbf{v}\times\tilde{\mathbf{H}}-\frac{\gamma}{\gamma+1}(\mathbf{v}\cdot\tilde{\mathbf{D}})\mathbf{v}, \tag{26c}$$

$$\tilde{\mathbf{H}}'=\gamma(\tilde{\mathbf{H}}-\mathbf{v}\times\tilde{\mathbf{D}})+\gamma^2\mathbf{v}\times(\tilde{\mathbf{D}}+\mathbf{v}\times\tilde{\mathbf{H}})-\frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\tilde{\mathbf{H}})\mathbf{v}, \tag{26d}$$

where primes denote quantities in the rotating frame, and $\gamma=1/\sqrt{1-\nu^2}$. Substituting Eqs. (26) into the constitutive equations given by Eqs. (13), and then solving for $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{H}}$, we find that the constitutive equations for the rotating medium, observed in the laboratory frame, are

$$\tilde{\mathbf{D}}=\frac{1}{(1-\nu^2)}\left(\left(\epsilon-\frac{\nu^2}{\mu}\right)\tilde{\mathbf{E}}+\left(\epsilon-\frac{1}{\mu}\right)[\mathbf{v}\times\tilde{\mathbf{B}}-(\mathbf{v}\cdot\tilde{\mathbf{E}})\mathbf{v}]\right), \tag{27a}$$

$$\tilde{\mathbf{H}}=\frac{1}{(1-\nu^2)}\left(\left(\frac{1}{\mu}-\epsilon\nu^2\right)\tilde{\mathbf{B}}+\left(\epsilon-\frac{1}{\mu}\right)[\mathbf{v}\times\tilde{\mathbf{E}}+(\mathbf{v}\cdot\tilde{\mathbf{B}})\mathbf{v}]\right). \tag{27b}$$

And upon rearranging Eqs. (27), we find that the resulting equations are identical to Minkowski's constitutive equations, first obtained in 1908 by using special relativity for the case of uniform motion:[1,2,4,5,11]

$$\tilde{\mathbf{D}}=\frac{1}{(1-\epsilon\mu\nu^2)}(\epsilon\tilde{\mathbf{E}}(1-\nu^2)+(\epsilon\mu-1)[\mathbf{v}\times\tilde{\mathbf{H}}-\epsilon(\mathbf{v}\cdot\tilde{\mathbf{E}})\mathbf{v}]), \tag{28a}$$

$$\tilde{\mathbf{B}}=\frac{1}{(1-\epsilon\mu\nu^2)}(\mu\tilde{\mathbf{H}}(1-\nu^2)-(\epsilon\mu-1)[\mathbf{v}\times\tilde{\mathbf{E}}-\mu(\mathbf{v}\cdot\tilde{\mathbf{H}})\mathbf{v}]). \tag{28b}$$

These constitutive equations are those used by observers in the laboratory frame that choose to work with one-form fields.

Also shown in the Appendix, vector fields in the rotating and lab frames are related according to

$$\mathbf{E}'=\gamma^2(\mathbf{E}+\mathbf{v}\times\mathbf{B})-\gamma^3\mathbf{v}\times(\mathbf{B}-\mathbf{v}\times\mathbf{E})-\frac{\gamma^3}{\gamma+1}(\mathbf{v}\cdot\mathbf{E})\mathbf{v}, \tag{29a}$$

$$\mathbf{B}'=\gamma(\mathbf{B}-\mathbf{v}\times\mathbf{E})-\frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\mathbf{B})\mathbf{v}, \tag{29b}$$

$$\mathbf{D}'=\gamma^2(\mathbf{D}+\mathbf{v}\times\mathbf{H})-\gamma^3\mathbf{v}\times(\mathbf{H}-\mathbf{v}\times\mathbf{D})-\frac{\gamma^3}{\gamma+1}(\mathbf{v}\cdot\mathbf{D})\mathbf{v}, \tag{29c}$$

$$\mathbf{H}'=\gamma(\mathbf{H}-\mathbf{v}\times\mathbf{D})-\frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\mathbf{H})\mathbf{v}. \tag{29d}$$

Following the same procedure used to obtain Eqs. (28), we substitute Eqs. (29) into Eqs. (16), and then solve for **D** and **B**. Carrying this out again leads directly to Minkowski's 1908 constitutive equations:[1,2,4,5,11]

$$\mathbf{D}=\frac{1}{(1-\epsilon\mu\nu^2)}(\epsilon\mathbf{E}(1-\nu^2)+(\epsilon\mu-1)[\mathbf{v}\times\mathbf{H}-\epsilon(\mathbf{v}\cdot\mathbf{E})\mathbf{v}]), \tag{30a}$$

$$\mathbf{B}=\frac{1}{(1-\epsilon\mu\nu^2)}(\mu\mathbf{H}(1-\nu^2)-(\epsilon\mu-1)[\mathbf{v}\times\mathbf{E}-\mu(\mathbf{v}\cdot\mathbf{H})\mathbf{v}]). \tag{30b}$$

These constitutive equations are used by those lab frame observers that work with vector fields, and are identical to the one-form constitutive equations given by Eqs. (28). Although different constitutive equations arise in the rotating frame, one-form fields and vector fields lead to constitutive equations of the same form in an inertial frame.

Field equations in the lab frame can be obtained by use of either Eqs. (28) or (30) in conjunction with Maxwell's field equations. Assuming static vector fields, for example, observers in the lab frame begin by writing Maxwell's field equations as

$$\nabla\cdot\mathbf{D}=4\pi\rho, \tag{31a}$$

$$\nabla\cdot\mathbf{B}=0, \tag{31b}$$

$$\nabla\times\mathbf{E}=0, \tag{31c}$$

$$\nabla\times\mathbf{H}=4\pi\mathbf{j}. \tag{31d}$$

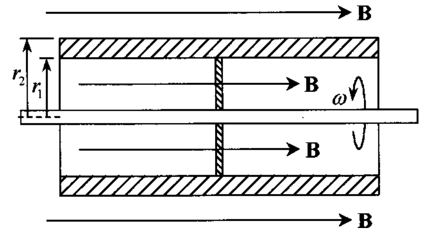

Fig. 1. A side cut-away view of a hollow cylinder of polarizable, permeable material that rotates within an external, axially directed magnetic field with velocity $\mathbf{v}=\omega r\mathbf{e}_\phi$ relative to the laboratory reference frame. Rotating and laboratory observers alike detect the presence of an electric field between inner and outer surfaces of the cylinder.

Then, by using Eqs. (30), lab frame observers can rewrite the field equations in the form[1]

$$\nabla\cdot\left(\frac{1}{(1-\nu^2)}\left[\left(\epsilon-\frac{\nu^2}{\mu}\right)\mathbf{E}+\left(\epsilon-\frac{1}{\mu}\right)[\mathbf{v}\times\mathbf{B}-(\mathbf{v}\cdot\mathbf{E})\mathbf{v}]\right]\right)=4\pi\rho, \tag{32a}$$

$$\nabla\cdot\mathbf{B}=0, \tag{32b}$$

$$\nabla\times\mathbf{E}=0, \tag{32c}$$

$$\nabla\times\left(\frac{1}{(1-\nu^2)}\left[\left(\frac{1}{\mu}-\epsilon\nu^2\right)\mathbf{B}+\left(\epsilon-\frac{1}{\mu}\right)[\mathbf{v}\times\mathbf{E}+(\mathbf{v}\cdot\mathbf{B})\mathbf{v}]\right]\right)=4\pi\mathbf{j}. \tag{32d}$$

An identical set of field equations is obtained when one-form fields are used in the lab frame. Observers in the lab frame can use either set of field equations to analyze experiments involving axial rotation so long as consistency is maintained throughout the analysis.

V. DERIVING THE FIELDS OBSERVED IN THE REFERENCE FRAME OF A ROTATING CYLINDER

As demonstrated in the preceding sections, when observers in a rotating frame define fields as one-forms, the form of the resulting field equations differs from those obtained by observers defining fields as vectors. In this section, we use both sets of field equations to derive the fields observed between inner and outer surfaces of a hollow cylinder with electric permittivity ϵ and magnetic permeability μ that rotates within an external, axially directed magnetic field.[1,6,12] Figure 1 shows a cut-away view of a such a cylinder. We assume that all fields are static, and that the cylinder is composed of a material that precludes free charges and currents.

Rotating observers choosing to express fields as one-forms begin by using the field equations[1,6]

$$\nabla\cdot\left(\epsilon\tilde{\mathbf{E}}'-\mathbf{v}\times\frac{1}{\mu}\tilde{\mathbf{B}}'\right)=0, \tag{33a}$$

$$\nabla\cdot\tilde{\mathbf{B}}'=0, \tag{33b}$$

$$\nabla\times\tilde{\mathbf{E}}'=0, \tag{33c}$$

$$\nabla\times\left(\frac{1}{\mu}(\tilde{\mathbf{B}}'-\mathbf{v}\times\tilde{\mathbf{E}}')\right)=0, \tag{33d}$$

where we have neglected terms of higher order than ν, and $\mathbf{v}=\omega r\mathbf{e}_\phi$ is the velocity of the rotating cylinder. Since the external magnetic field is parallel to the axis of the cylinder, Eq. (33a) implies that the normal component of $\epsilon\tilde{\mathbf{E}}' - \mathbf{v} \times (1/\mu)\tilde{\mathbf{B}}'$ is continuous at the surface of the rotating medium. Assuming that the cylinder is sufficiently long for end effects to be ignored, Eq. (33a) then implies that

$$\epsilon_{\mathrm{IN}}\tilde{\mathbf{E}}'_{\mathrm{IN}} - \mathbf{v}\times\frac{1}{\mu_{\mathrm{IN}}}\tilde{\mathbf{B}}'_{\mathrm{IN}} = \tilde{\mathbf{E}}'_{\mathrm{OUT}} - \mathbf{v}\times\tilde{\mathbf{B}}'_{\mathrm{OUT}}, \tag{34}$$

where the subscripts denote quantities taken inside or outside the material, and we have taken ϵ_{OUT} and μ_{OUT} equal to unity. Outside the material, the fields in the lab frame are $\tilde{\mathbf{E}}_{\mathrm{OUT}} = \tilde{\mathbf{0}}$ and $\tilde{\mathbf{B}}_{\mathrm{OUT}} \neq \tilde{\mathbf{0}}$. Using these fields in Eqs. (26a) and (26b) gives the external fields in the rotating frame as

$$\tilde{\mathbf{E}}'_{\mathrm{OUT}} = \mathbf{v}\times\tilde{\mathbf{B}}_{\mathrm{OUT}}, \tag{35a}$$

$$\tilde{\mathbf{B}}'_{\mathrm{OUT}} = \tilde{\mathbf{B}}_{\mathrm{OUT}}. \tag{35b}$$

Thus, the right-hand side of Eq. (34) is zero. Simplifying a bit, Eq. (34) then leads to

$$\tilde{\mathbf{E}}'_{\mathrm{IN}} = \frac{1}{\epsilon\mu}\mathbf{v}\times\tilde{\mathbf{B}}'_{\mathrm{IN}}, \tag{36}$$

where we have dropped the subscripts from ϵ_{IN} and μ_{IN}. Again taking into consideration that the magnetic field is parallel to the axis of rotation, Eq. (33d) implies that $\tilde{\mathbf{B}}'_{\mathrm{IN}} = \mu\tilde{\mathbf{B}}'_{\mathrm{OUT}}$. Substituting this relationship into Eq. (36) gives

$$\tilde{\mathbf{E}}'_{\mathrm{IN}} = \frac{1}{\epsilon}\mathbf{v}\times\tilde{\mathbf{B}}'_{\mathrm{OUT}}. \tag{37}$$

This is the electric field, between inner and outer surfaces of the cylinder, that is detected by observers in the rotating frame that work with one-form fields. And upon using Eq. (37) in Eq. (26a), we find that observers in the lab frame detect an electric field of the form[1,6]

$$\tilde{\mathbf{E}}_{\mathrm{IN}} = \left(\frac{1}{\epsilon} - \mu\right)\mathbf{v}\times\tilde{\mathbf{B}}_{\mathrm{OUT}}. \tag{38}$$

On the other hand, when rotating observers work with vectors, the field equations assume the form

$$\nabla\cdot\left(\epsilon\mathbf{E}' + \left(\epsilon - \frac{1}{\mu}\right)\mathbf{v}\times\mathbf{B}'\right) = 0, \tag{39a}$$

$$\nabla\cdot(\mathbf{B}' + \mathbf{v}\times\mathbf{E}') = 0, \tag{39b}$$

$$\nabla\times(\mathbf{E}' + \mathbf{v}\times\mathbf{B}') = 0, \tag{39c}$$

$$\nabla\times\frac{1}{\mu}\mathbf{B}' = 0, \tag{39d}$$

where as before we have neglected terms of order higher than ν. According to Eq. (39a), when the external magnetic field is parallel to the axis of rotation, rotating observers can write

$$\epsilon_{\mathrm{IN}}\mathbf{E}'_{\mathrm{IN}} + \left(\epsilon_{\mathrm{IN}} - \frac{1}{\mu_{\mathrm{IN}}}\right)\mathbf{v}\times\mathbf{B}'_{\mathrm{IN}} = \mathbf{E}'_{\mathrm{OUT}}. \tag{40}$$

Taking the external fields in the lab frame to be $\mathbf{E}_{\mathrm{OUT}} = \mathbf{0}$ and $\mathbf{B}_{\mathrm{OUT}} \neq \mathbf{0}$, Eqs. (29a) and (29b) imply that the external fields in the rotating frame are

$$\mathbf{E}'_{\mathrm{OUT}} = \mathbf{0}, \tag{41a}$$

$$\mathbf{B}'_{\mathrm{OUT}} = \mathbf{B}_{\mathrm{OUT}}. \tag{41b}$$

Substituting these fields into Eq. (40), and noting that Eq. (39d) implies that $\mathbf{B}'_{\mathrm{IN}} = \mu\mathbf{B}'_{\mathrm{OUT}}$ for the case of an axially directed magnetic field, we find that

$$\mathbf{E}'_{\mathrm{IN}} = \left(\frac{1}{\epsilon} - \mu\right)\mathbf{v}\times\mathbf{B}'_{\mathrm{OUT}}. \tag{42}$$

This electric field is the one detected by rotating observers that choose to express fields as vectors. Using Eq. (42) in Eq. (29a), we find that observers in the lab frame detect an electric field identical to that given by Eq. (38):[1,6]

$$\mathbf{E}_{\mathrm{IN}} = \left(\frac{1}{\epsilon} - \mu\right)\mathbf{v}\times\mathbf{B}_{\mathrm{OUT}}. \tag{43}$$

As expected, one-form and vector fields assume different forms in the rotating frame, but take on the same form when transformed to the inertial frame of the laboratory. Therefore, field equations obtained on the basis of one-form fields and those obtained on the basis of vector fields both lead to predictions that are consistent with known experimental results.[4,6,11,12]

VI. CONCLUSIONS

We have shown two methods by which Maxwell's equations can be applied to rotating linear media. In the first method, the covariant electromagnetic tensor $F_{\alpha\beta}$ was used; and in the second method, the contravariant tensor $F^{\alpha\beta}$ was used. We began by explaining that whether one works with a covariant or contravariant tensor is dictated by whether one chooses to express fields in terms of a vector basis or in terms of a dual one-form basis. Using this formalism, general field equations were derived in terms of both vector fields and one-form fields in the rotating and laboratory frames. Fields in the presence of matter were then related to those in a vacuum by using the covariant form of Minkowski's constitutive equations,[2–5] generalized to noninertial frames. Next, the constitutive equations were transformed from the rotating frame to the lab frame. Carrying this out, we found that although vector and one-form constitutive equations differ in the rotating frame, in the lab frame both pairs of constitutive equations are in agreement with the 1908 constitutive equations of Minkowski.[1,2,4,5,11]

We then demonstrated the use of vector and one-form field equations in a rotating frame. Both sets of field equations were used to derive the fields observed in the reference frame of a polarizable, permeable cylinder that rotates within an axially directed magnetic field.[1,6,12] As expected, we found that vector and one-form fields take on different forms in the rotating frame, but assume the same form in the inertial frame of the laboratory.

We conclude that when applying the covariant form of Maxwell's equations to a noninertial frame, the choice between working with a covariant or contravariant electromagnetic tensor depends upon whether one chooses to express fields in terms of a vector basis or in terms of a dual one-form basis. More particularly, we conclude that the electric and magnetic fields can be expressed either as vectors or as one-forms so long as one is consistent throughout the analysis.

APPENDIX: DERIVING A DIRECT FIELD TRANSFORMATION FROM THE LABORATORY FRAME TO THE ROTATING FRAME

We wish to derive a transformation that relates fields observed in a rotating reference frame to those observed in a laboratory frame. The metric tensor in the rotating coordinate system is[6,7,19]

$$g'_{\alpha\beta}=\begin{pmatrix} 1-\nu^2 & -\nu_x & -\nu_y & 0 \\ -\nu_x & -1 & 0 & 0 \\ -\nu_y & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A1}$$

where we have taken $\nu_x=-\omega y'$ and $\nu_y=\omega x'$, and $\nu^2=\nu_x^2+\nu_y^2$. The inverse metric tensor in rotating coordinates is[7]

$$g'^{\alpha\beta}=\begin{pmatrix} 1 & -\nu_x & -\nu_y & 0 \\ -\nu_x & -1+\nu_x^2 & \nu_x\nu_y & 0 \\ -\nu_y & \nu_x\nu_y & -1+\nu_y^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A2}$$

We begin by noting that, although the rotating frame has a velocity $\mathbf{v}=\omega r\mathbf{e}_\phi$ relative to the lab frame, at any given instant a momentarily comoving reference frame (MCRF) of an observer at rest in the rotating frame has a uniform velocity $\mathbf{v}$ relative to the lab frame. Thus, we can obtain a relationship between fields in the rotating and lab frames as follows. We first find a relationship between fields in the rotating frame and those in the MCRF, and then we use a Lorentz transformation to relate fields in the MCRF to those in the lab frame. Upon eliminating MCRF quantities between the two transformations, we find a direct transformation from the lab frame to the rotating frame.

The covariant electromagnetic tensor $F_{\alpha\beta}$ can be transformed from the rotating frame to the MCRF by using

$$F''=R^TF'R, \tag{A3}$$

where F'' is the electromagnetic tensor in the MCRF, R^T is the transpose of R, and R is a transformation that relates quantities in the rotating frame to those in the MCRF, given by[1,20]

$$R^\alpha_\beta=\frac{\partial x'^\alpha}{\partial x''^\beta}=\begin{pmatrix} \gamma & \nu_x\gamma^2 & \nu_y\gamma^2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A4}$$

in which $\gamma=1/\sqrt{1-\nu^2}$. Similarly, the electromagnetic tensor can be transformed from the lab frame to the MCRF by using

$$F''=L^TFL, \tag{A5}$$

where F is the electromagnetic tensor in the lab frame, and L is the well-known Lorentz transformation that connects two frames moving with respect to each other in an arbitrary direction in the $x-y$ plane,[21]

$$L^\alpha_\beta=\frac{\partial x^\alpha}{\partial x''^\beta}=\begin{pmatrix} \gamma & \nu_x\gamma & \nu_y\gamma & 0 \\ \nu_x\gamma & 1+\nu_x^2A & \nu_x\nu_yA & 0 \\ \nu_y\gamma & \nu_x\nu_yA & 1+\nu_y^2A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A6}$$

in which $A=(\gamma-1)/\nu^2$. Equating Eqs. (A3) and (A5) and solving for F' leads to an expression that transforms quantities from the lab frame directly to the rotating frame:

$$F'=(LR^{-1})^TF(LR^{-1}). \tag{A7}$$

Working out the matrix multiplication, we find that Eq. (A7) can be rewritten simply as

$$F'=N^TFN, \tag{A8}$$

where N is given by

$$N^\alpha_\beta=\begin{pmatrix} 1 & \nu_x(\gamma-\gamma^2) & \nu_y(\gamma-\gamma^2) & 0 \\ \nu_x & 1+\nu_x^2(A-\gamma^2) & \nu_x\nu_y(A-\gamma^2) & 0 \\ \nu_y & \nu_x\nu_y(A-\gamma^2) & 1+\nu_y^2(A-\gamma^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A9}$$

Upon inserting the covariant electromagnetic tensor,

$$F_{\alpha\beta}=\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{A10}$$

into the right-hand side of Eq. (A8), and noting that the components of $F_{\alpha\beta}$ are the components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$, we find that one-form fields in the rotating frame are related to those in the lab frame by

$$\tilde{\mathbf{E}}'=\tilde{\mathbf{E}}+\mathbf{v}\times\tilde{\mathbf{B}}-\frac{\gamma}{\gamma+1}(\mathbf{v}\cdot\tilde{\mathbf{E}})\mathbf{v}, \tag{A11a}$$

$$\tilde{\mathbf{B}}'=\gamma(\tilde{\mathbf{B}}-\mathbf{v}\times\tilde{\mathbf{E}})+\gamma^2\mathbf{v}\times(\tilde{\mathbf{E}}+\mathbf{v}\times\tilde{\mathbf{B}})-\frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\tilde{\mathbf{B}})\mathbf{v}, \tag{A11b}$$

where quantities in the rotating frame carry a prime. Carrying out the same procedure used to find Eqs. (A11), analogous transformations for auxiliary fields $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{H}}$ are obtained:

$$\tilde{\mathbf{D}}'=\tilde{\mathbf{D}}+\mathbf{v}\times\tilde{\mathbf{H}}-\frac{\gamma}{\gamma+1}(\mathbf{v}\cdot\tilde{\mathbf{D}})\mathbf{v}, \tag{A12a}$$

$$\tilde{\mathbf{H}}'=\gamma(\tilde{\mathbf{H}}-\mathbf{v}\times\tilde{\mathbf{D}})+\gamma^2\mathbf{v}\times(\tilde{\mathbf{D}}+\mathbf{v}\times\tilde{\mathbf{H}})-\frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\tilde{\mathbf{H}})\mathbf{v}. \tag{A12b}$$

With the transformation for the covariant electromagnetic tensor known, the transformation for the contravariant tensor $F^{\alpha\beta}$ can be most simply obtained by taking

$$F'=MFM^T, \tag{A13}$$

where F' now represents the contravariant electromagnetic tensor in the rotating frame, and $M=N^{-1}$.[6] Taking the inverse of Eq. (A9), we find that M is given by

$$M^{\alpha}_{\beta}=\begin{pmatrix} \gamma^2(1-\nu^2\gamma) & -\nu_x\gamma(\gamma-\gamma^2) & -\nu_y\gamma(\gamma-\gamma^2) & 0 \\ -\nu_x\gamma & 1-\nu_x^2\gamma(A-\gamma) & -\nu_x\nu_y\gamma(A-\gamma) & 0 \\ -\nu_y\gamma & -\nu_x\nu_y\gamma(A-\gamma) & 1-\nu_y^2\gamma(A-\gamma) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A14}$$

Using the contravariant electromagnetic tensor,

$$F^{\alpha\beta}=\begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{A15}$$

in the right-hand side of Eq. (A13), we find that vector fields in the rotating and lab frames are related according to

$$\mathbf{E}'=\gamma^2(\mathbf{E}+\mathbf{v}\times\mathbf{B})-\gamma^3\mathbf{v}\times(\mathbf{B}-\mathbf{v}\times\mathbf{E})-\frac{\gamma^3}{\gamma+1}(\mathbf{v}\cdot\mathbf{E})\mathbf{v}, \tag{A16a}$$

$$\mathbf{B}'=\gamma(\mathbf{B}-\mathbf{v}\times\mathbf{E})-\frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\mathbf{B})\mathbf{v}. \tag{A16b}$$

A similar transformation is easily obtained for auxiliary fields **D** and **H**:

$$\mathbf{D}'=\gamma^2(\mathbf{D}+\mathbf{v}\times\mathbf{H})-\gamma^3\mathbf{v}\times(\mathbf{H}-\mathbf{v}\times\mathbf{D})-\frac{\gamma^3}{\gamma+1}(\mathbf{v}\cdot\mathbf{D})\mathbf{v}, \tag{A17a}$$

$$\mathbf{H}'=\gamma(\mathbf{H}-\mathbf{v}\times\mathbf{D})-\frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\mathbf{H})\mathbf{v}. \tag{A17b}$$

We note that when the speed of rotation is taken to be very much less than the speed of light, the relationship between one-form and vector fields in the rotating frame and those in the lab frame simplifies to[13,22]

$$\tilde{\mathbf{E}}'=\tilde{\mathbf{E}}+\mathbf{v}\times\tilde{\mathbf{B}}, \tag{A18a}$$

$$\tilde{\mathbf{B}}'=\tilde{\mathbf{B}}, \tag{A18b}$$

$$\mathbf{E}'=\mathbf{E}, \tag{A19a}$$

$$\mathbf{B}'=\mathbf{B}-\mathbf{v}\times\mathbf{E}. \tag{A19b}$$

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NEVER ECONOMIZE

One way to lessen the risk inherent in undertaking a major project is to make sure that you spend enough money on it. After a research department or funding agency has invested enough in your goals, it has a real stake in your success and becomes very reluctant to admit that your project is not working out. No one ever got ahead in science by saving money.

Peter J. Feibelman, *A Ph.D. Is Not Enough—A Guide to Survival in Science* (Addison–Wesley, Reading, MA, 1993), p. 105.