Applying relativistic electrodynamics to a rotating material medium

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We apply relativistic electrodynamics to a rotating linear medium. Covariant field equations are used to derive general field equations in a rotating coordinate system. We argue that the relation between fields in the presence of matter and those in a vacuum is necessarily dependent upon the coordinate system used. Constitutive equations are then derived in the rotating and laboratory reference frames. We find that our constitutive equations in the laboratory frame agree with Minkowski's constitutive equations, derived on the basis of special relativity in 1908. Thus we conclude that special relativity can be used in the analysis of experiments involving rotational motion. To exemplify the use of special relativity, we derive an experimentally observed result of a 1913 experiment performed by Wilson and Wilson in which a polarizable, permeable cylinder was rotated in a uniform, axially directed magnetic field.

I. INTRODUCTION

For more than 80 years, the compatibility of special relativity and classical electrodynamics has been generally accepted; however, there still exists one area of the subject which apparently leads to some confusion. How does one apply relativistic electrodynamics to a material medium? More particularly, how does one deal with rotating material media? There is no ambiguity when the relative motion is uniform; however, as pointed out in a recent article by Pellegrini and Swift,[1] rotational motion of a material medium leads to considerable confusion. This paper addresses the simplest method by which relativistic electrodynamics can be applied to a rotating material medium.

One important point to recognize is that a rotating reference frame is not a truly inertial frame, due to the presence of inertial forces therein. The application of Maxwell's equations to accelerating frames is straightforward, being dictated by the principle of general covariance coupled with the equivalence principle.[1–4] When matter is present, however, the response of the material must also be considered. Fields in the presence of an isotropic, linear material are related to fields **E** and **B** by the constitutive equations $\mathbf{D}=\epsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$, where the fields **D** and **H** are the electric and magnetic fields in the presence of a material with electric permittivity ϵ and magnetic permeability μ. We have no reason to presuppose that these constitutive equations should be invariant; in fact, the constitutive equations are known to be frame dependent.[5–8] It is our opinion that the confusion surrounding rotating material media arises from a failure to recognize the frame dependence of the constitutive equations.

In the next section we use a covariant form of Maxwell's equations in rotating coordinates to derive three-dimensional field equations for the rotating reference frame. We start by noting that the covariant derivative can be written as an ordinary partial derivative in rotating coordinates. We then obtain general field equations in three-dimensional notation, expressed in terms of the fields in the presence of matter. We discuss the need for constitutive equations with which to relate fields in matter to those in a vacuum.

In Sec. III we introduce Minkowski's covariant constitutive equations for uniform motion,[6,8–10] and show how they are used for the case of uniformly moving material media observed in an inertial frame of reference. We then explain that in order to use Minkowski's covariant constitutive equations in a noninertial frame, one must simply raise and lower tensor indices with the metric tensor in arbitrary coordinates. Adhering to this familiar rule, we then derive constitutive equations in the rotating coordinate system. Upon substituting these constitutive equations into the field equations, derived in Sec. II, we obtain field equations for the rotating frame. When the rotational velocity is much smaller than that of light, the field equations assume a form known to agree with all experimental results to date.[1] In addition, we introduce covariant equations for the polarization and magnetization, which are then used to determine the polarization and magnetization in the rotating reference frame. And finally, we introduce and discuss an alternative derivation of the field equations, utilizing the polarization and magnetization.

Section IV is devoted to the laboratory reference frame. We introduce two methods by which constitutive equations for a rotating material can be derived in the lab frame. With the first method, one uses field transformations to transform the constitutive equations from the rotating frame to the lab frame. The second method of derivation consists of expressing Minkowski's covariant constitutive equations[6,8–10] in the lab frame, and using the four-velocity of the rotating material. Both methods produce the same end result: constitutive equations which agree with those first derived by Minkowski in 1908, using special relativity.[6,9–11] On the basis of this agreement, we come to the conclusion that special relativity can be used to analyze experiments involving rotational motion.

In Sec. V, we demonstrate the application of special relativity to rotational motion by deriving an experimentally observed result of a 1913 experiment of Wilson and Wilson.[1,11,12] The Wilsons measured an electric potential between the inner and outer surfaces of a polarizable, permeable cylinder set in rotation within a uniform, axially directed magnetic field. The measured electric potential closely agreed with a potential predicted by special relativity.[1,9,11] We derive this electric potential, using Maxwell's equations in conjunction with the constitutive equations, in the lab frame.

We begin Sec. VI by considering a field transformation that relates the polarization and magnetization in the rotating frame to the polarization and magnetization observed in the lab frame. We then obtain the polarization and magnetization in the lab frame, written in terms of fields observed in the rotating frame. The polarization and magnetization in the lab frame are then used to analyze two well-known experiments. First, we consider the case in which a polarizable cylinder rotates in the lab frame, in an external, axially directed electric field. Next, we consider the case in which a magnetizable cylinder rotates in the lab frame, in an external, axially directed magnetic field. In each case, we find a result that agrees with the predictions of special relativity.[10]

In the Appendix we derive field transformations that relate quantities in the rotating frame to those in the lab frame. We begin by deriving a transformation that relates fields in the rotating frame to those in a momentarily comoving reference frame (MCRF) of an observer at rest in the rotating frame. Fields in the MCRF are then related to those in the lab frame by use of well-known special relativity field transformations.[13] Upon eliminating MCRF quantities between the two transformations, we are left with a transformation relating fields in the rotating frame to fields in the lab frame.

II. FIELD EQUATIONS IN A ROTATING FRAME

Although rotation does not lead to space–time curvature, a rotating reference frame is not a truly inertial frame of reference due to the presence of inertial forces.[1,4,14–16] Maxwell's equations can be extended to encompass accelerating reference frames by using the covariant field equations in arbitrary coordinates:[2]

$$\nabla_{\alpha}F^{\alpha\beta}=4\pi j^{\beta}, \tag{1a}$$

$$\epsilon^{\mu\nu\kappa\lambda}\nabla_{\nu}F_{\kappa\lambda}=0, \tag{1b}$$

where source charges and currents are given by the current four-vector j^{β} and we have used, and will continue to use, units in which the speed of light is set equal to unity: $c=1$. We start by choosing to work with a covariant form of the electromagnetic field tensor $F_{\alpha\beta}$,[17] defined as

$$F_{\alpha\beta}=\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{2}$$

The contravariant electromagnetic tensor is then obtained simply by raising indices with the inverse metric tensor:

$$F^{\alpha\beta}=g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}. \tag{3}$$

When written in terms of the covariant electromagnetic tensor, Eq. (1a) assumes the form

$$\nabla_{\alpha}(g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu})=4\pi j^{\beta}. \tag{4}$$

When the field equations are to take the presence of matter into account, we introduce another covariant electromagnetic tensor, composed of the components of the electric and magnetic fields in the presence of matter.[18] This tensor is defined as

$$H_{\alpha\beta}=\begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & -H_z & H_y \\ -D_y & H_z & 0 & -H_x \\ -D_z & -H_y & H_x & 0 \end{pmatrix}, \tag{5}$$

where **D** and **H** are the electric displacement and magnetic field, respectively. Using this electromagnetic tensor, the field equations take on the form

$$\nabla_\alpha(g^{\alpha\mu}g^{\beta\nu}H_{\mu\nu})=4\pi j^\beta, \tag{6a}$$

$$\epsilon^{\mu\nu\kappa\lambda}\nabla_\nu F_{\kappa\lambda}=0, \tag{6b}$$

where free charges and currents are given by the current four-vector j^β.

We wish to write Eqs. (6) in three-dimensional notation, but before doing so we take note of a great simplification: In rotating coordinates, the covariant derivative is exactly equal to the ordinary partial derivative.[19] Taking this into consideration, the field equations assume the form

$$\partial_\alpha(g^{\alpha\mu}g^{\beta\nu}H_{\mu\nu})=4\pi j^\beta, \tag{7a}$$

$$\epsilon^{\mu\nu\kappa\lambda}\partial_\nu F_{\kappa\lambda}=0. \tag{7b}$$

Using the metric tensor in rotating coordinates, given in the Appendix, and limiting ourselves to the case of static fields, the field equations become

$$\nabla\cdot(\mathbf{D}'-\mathbf{v}\times\mathbf{H}')=4\pi\rho', \tag{8a}$$

$$\nabla\cdot\mathbf{B}'=0, \tag{8b}$$

$$\nabla\times\mathbf{E}'=0, \tag{8c}$$

$$\nabla\times[\mathbf{H}'-\mathbf{v}\times\mathbf{D}'+\mathbf{v}\times(\mathbf{v}\times\mathbf{H}')]=4\pi\mathbf{j}', \tag{8d}$$

where quantities in the rotating frame carry a prime, and observers at rest in the rotating frame have velocity $\mathbf{v}=\omega r\hat{\mathbf{e}}_\phi$ relative to the laboratory frame. Equations (8) are the general field equations for the rotating reference frame, written in three-dimensional notation.

As yet, we have not specified a relationship between the auxiliary fields, **D** and **H**, and fundamental fields, **E** and **B**. Recall that we defined tensors $F_{\alpha\beta}$ and $H_{\alpha\beta}$ in the rotating reference frame, but we said nothing of the relationship between fields. Thus, while the tensor $F_{\alpha\beta}$ is well understood to comprise the components of the fundamental electric and magnetic fields, the tensor $H_{\alpha\beta}$ is merely a definition at this point. We can relate auxiliary fields in the presence of matter to the fundamental fields with constitutive equations of the general form[13]

$$\mathbf{D}=\mathbf{D}[\mathbf{E},\mathbf{B}], \tag{9a}$$

$$\mathbf{H}=\mathbf{H}[\mathbf{E},\mathbf{B}], \tag{9b}$$

where the square brackets are used to indicate that the relationship may not necessarily be simple, and in fact may depend upon the past history of the material. However, the question arises: What form do the constitutive equations take in a noninertial frame?

III. CONSTITUTIVE EQUATIONS IN ROTATING COORDINATES

As mentioned in the last section, auxiliary fields in the presence of matter can be related to the fundamental fields by use of constitutive equations in the rotating reference frame. We start by considering a covariant form of the constitutive equations, first introduced by Minkowski in 1908.[6,9,10]

$$H_{\lambda\mu}u^\mu=\epsilon F_{\lambda\mu}u^\mu, \tag{10a}$$

$$\epsilon^{\sigma\lambda\mu\nu}F_{\lambda\mu}u_\nu=\mu\epsilon^{\sigma\lambda\mu\nu}H_{\lambda\mu}u_\nu, \tag{10b}$$

where $\epsilon^{\sigma\lambda\mu\nu}$ is the fourth-rank Levi–Civita antisymmetric tensor with $\epsilon^{0123}=+1$, and electric permittivity ϵ and magnetic permeability μ are proper quantities defined in the local rest frame of the material. When Eqs. (10) are used for the special relativistic case of uniformly moving linear media observed from an inertial frame of reference, we take the four-velocity of the moving material to be $u^\mu=\gamma(1,\mathbf{v})$, where $\gamma=1/\sqrt{1-v^2}$. Upon using the electromagnetic tensors given by Eqs. (2) and (5), Eqs. (10) then lead to the general result

$$\mathbf{D}+\mathbf{v}\times\mathbf{H}=\epsilon(\mathbf{E}+\mathbf{v}\times\mathbf{B}), \tag{11a}$$

$$\mathbf{B}+\mathbf{E}\times\mathbf{v}=\mu(\mathbf{H}+\mathbf{D}\times\mathbf{v}), \tag{11b}$$

where **v** is the uniform velocity of the moving material relative to the inertial frame. Solving Eqs. (11) for **D** and **B** leads directly to

$$\mathbf{D}=\frac{1}{(1-\epsilon\mu v^2)}\{\epsilon\mathbf{E}(1-v^2)+(\epsilon\mu-1)\times[\mathbf{v}\times\mathbf{H}-\epsilon(\mathbf{v}\cdot\mathbf{E})\mathbf{v}]\}, \tag{12a}$$

$$\mathbf{B}=\frac{1}{(1-\epsilon\mu v^2)}\{\mu\mathbf{H}(1-v^2)-(\epsilon\mu-1)\times[\mathbf{v}\times\mathbf{E}-\mu(\mathbf{v}\cdot\mathbf{H})\mathbf{v}]\}. \tag{12b}$$

Equations (12) were first introduced in 1908 by Minkowski for the special relativistic case of uniformly moving linear media observed from an inertial frame. Thus Eqs. (12) are referred to as Minkowski's constitutive equations.[6,9–11]

In order to obtain constitutive equations in the rotating frame, we need to generalize Eqs. (10) to noninertial frames of reference. Such a generalization can most simply be carried out by recognizing that in arbitrary coordinates, tensor indices must be raised and lowered with the metric tensor. With this in mind, the constitutive equations may be written as

$$H_{\lambda\mu}u^\mu=\epsilon F_{\lambda\mu}u^\mu, \tag{13a}$$

$$\epsilon^{\sigma\lambda\mu\nu}F_{\lambda\mu}g_{\nu\alpha}u^\alpha=\mu\epsilon^{\sigma\lambda\mu\nu}H_{\lambda\mu}g_{\nu\alpha}u^\alpha. \tag{13b}$$

Thus we can find constitutive equations in any coordinate system, so long as we know the components of the four-velocity u^μ.

We wish to apply Eqs. (13) to a rotating frame and obtain the constitutive equations therein, expressed in three-dimensional notation. An observer at rest in the rotating frame has a four-velocity $u^\mu=\gamma(1,0,0,0)$, where, as before, $\gamma=1/\sqrt{1-v^2}$. With this four-velocity, Eqs. (13) become

$$H_{\lambda 0}=\epsilon F_{\lambda 0}, \tag{14a}$$

$$\epsilon^{\sigma\lambda\mu\nu}F_{\lambda\mu}g_{\nu 0}=\mu\epsilon^{\sigma\lambda\mu\nu}H_{\lambda\mu}g_{\nu 0}. \tag{14b}$$

Using the metric tensor for rotating coordinates, given in the Appendix, we find that observers at rest in the rotating frame must use the relations

$$\mathbf{D}'=\epsilon\mathbf{E}', \tag{15a}$$

$$(1-v^2)\mathbf{B}'+\mathbf{E}'\times\mathbf{v}=\mu[(1-v^2)\mathbf{H}'+\mathbf{D}'\times\mathbf{v}], \tag{15b}$$

where observers at rest in the rotating frame have a velocity $\mathbf{v}=\omega r\hat{\mathbf{e}}_\phi$ relative to the laboratory reference frame. Solving

Eqs. (15) for $\mathbf{D}'$ and $\mathbf{H}'$, we find that the constitutive equations in the rotating frame are

$$\mathbf{D}' = \epsilon \mathbf{E}', \tag{16a}$$

$$\mathbf{H}' = \frac{1}{\mu}\,\mathbf{B}' + \frac{1}{(1-v^2)}\left(\epsilon - \frac{1}{\mu}\right)\mathbf{v}\times\mathbf{E}'. \tag{16b}$$

Now we have the desired relationship between auxiliary and fundamental fields in the rotating reference frame. Simple substitution of Eqs. (16) into the field equations leads directly to

$$\nabla\cdot\left(\epsilon\mathbf{E}' - \mathbf{v}\times\frac{1}{\mu}\,\mathbf{B}' - \frac{1}{(1-v^2)}\left(\epsilon - \frac{1}{\mu}\right)\mathbf{v}\times(\mathbf{v}\times\mathbf{E}')\right) = 4\pi\rho', \tag{17a}$$

$$\nabla\cdot\mathbf{B}' = 0, \tag{17b}$$

$$\nabla\times\mathbf{E}' = 0, \tag{17c}$$

$$\nabla\times\left\{\frac{1}{\mu}\,[\mathbf{B}' - \mathbf{v}\times\mathbf{E}' + \mathbf{v}\times(\mathbf{v}\times\mathbf{B}')]\right\} = 4\pi\mathbf{j}'. \tag{17d}$$

These field equations are those used by observers at rest in the rotating frame. When the speed of rotation is taken to be much smaller than that of light, Eqs. (17) simplify to a form known to agree with all experimental results to date:[1]

$$\nabla\cdot\left(\epsilon\mathbf{E}' - \mathbf{v}\times\frac{1}{\mu}\,\mathbf{B}'\right) = 4\pi\rho', \tag{18a}$$

$$\nabla\cdot\mathbf{B}' = 0, \tag{18b}$$

$$\nabla\times\mathbf{E}' = 0, \tag{18c}$$

$$\nabla\times\left\{\frac{1}{\mu}\,[\mathbf{B}' - \mathbf{v}\times\mathbf{E}']\right\} = 4\pi\mathbf{j}'. \tag{18d}$$

At this point, we should mention that rotating observers could just as easily obtain Eqs. (17) by using the polarization $\mathbf{P}'$ and magnetization $\mathbf{M}'$. Since we have already committed ourselves to the covariant convention by choosing to work with covariant electromagnetic tensors $H_{\alpha\beta}$ and $F_{\alpha\beta}$, we continue by choosing a definition of the electromagnetic tensor $H_{\alpha\beta}$ given by

$$H_{\alpha\beta} = F_{\alpha\beta} - 4\pi M_{\alpha\beta}, \tag{19}$$

where the tensor $M_{\alpha\beta}$ is a magnetization four-tensor.[10,13] Insisting that auxiliary fields $\mathbf{D}'$ and $\mathbf{H}'$ are defined in the rotating frame according to

$$\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}', \tag{20a}$$

$$\mathbf{H}' = \mathbf{B}' - 4\pi\mathbf{M}', \tag{20b}$$

we are led directly to a covariant magnetization tensor composed of the vector components of the polarization and magnetization:[10,13,20]

$$M_{\alpha\beta} = \begin{pmatrix} 0 & -P_x & -P_y & -P_z \\ P_x & 0 & -M_z & M_y \\ P_y & M_z & 0 & -M_x \\ P_z & -M_y & M_x & 0 \end{pmatrix}. \tag{21}$$

Substituting Eq. (19) into each of Eqs. (10), and then solving for $M_{\alpha\beta}$, we find that the covariant equations for the polarization and magnetization are[9,21]

$$M_{\lambda\mu}u^{\mu} = \frac{1}{4\pi}\,(\epsilon - 1)F_{\mu\lambda}u^{\mu}, \tag{22a}$$

$$\epsilon^{\sigma\lambda\mu\nu}M_{\lambda\mu}u_{\nu} = \frac{1}{4\pi}\left(1 - \frac{1}{\mu}\right)\epsilon^{\sigma\lambda\mu\nu}F_{\lambda\mu}u_{\nu}. \tag{22b}$$

With a knowledge of the four-velocity, we can use Eqs. (22) to find the polarization and magnetization in any reference frame we choose.

Following the same steps as those used in the derivation of the constitutive equations, we find that the polarization and magnetization in the rotating frame are

$$\mathbf{P}' = \frac{1}{4\pi}\,(\epsilon - 1)\mathbf{E}', \tag{23a}$$

$$\mathbf{M}' = \frac{1}{4\pi}\left(1 - \frac{1}{\mu}\right)\mathbf{B}' + \frac{1}{4\pi(1-v^2)}\left(\frac{1}{\mu} - \epsilon\right)\mathbf{v}\times\mathbf{E}'. \tag{23b}$$

To use Eqs. (23) in the field equations, we merely recall that the definitions of the fields $\mathbf{D}'$ and $\mathbf{H}'$ are given by Eqs. (20). Upon substituting Eqs. (23) into Eqs. (20) we find a pair of equations identical to Eqs. (16), the constitutive equations in the rotating reference frame. Hence, one finds field equations that are identical to Eqs. (17) in the rotating reference frame.

IV. THE LABORATORY REFERENCE FRAME

In the preceding section, all calculations were performed in the rotating reference frame. Now we transform the constitutive equations from the rotating frame to the laboratory frame. As shown in the Appendix, fields in the rotating frame are related to those observed in the lab frame by field transformations:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{\gamma}{\gamma+1}\,(\mathbf{v}\cdot\mathbf{E})\mathbf{v}, \tag{24a}$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}) + \gamma^2\mathbf{v}\times(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - \frac{\gamma^2}{\gamma+1}\,(\mathbf{v}\cdot\mathbf{B})\mathbf{v}, \tag{24b}$$

where the prime denotes quantities in the rotating frame, and $\gamma = 1/\sqrt{1-v^2}$. Using the same procedure used to obtain Eqs. (24), similar transformations are found for the auxiliary fields:

$$\mathbf{D}' = \mathbf{D} + \mathbf{v}\times\mathbf{H} - \frac{\gamma}{\gamma+1}\,(\mathbf{v}\cdot\mathbf{D})\mathbf{v}, \tag{25a}$$

$$\mathbf{H}' = \gamma(\mathbf{H} - \mathbf{v}\times\mathbf{D}) + \gamma^2\mathbf{v}\times(\mathbf{D} + \mathbf{v}\times\mathbf{H}) - \frac{\gamma^2}{\gamma+1}\,(\mathbf{v}\cdot\mathbf{B})\mathbf{v}. \tag{25b}$$

Simply by substituting Eqs. (24) and Eqs. (25) into Eqs. (16), and then solving the resulting expressions for $\mathbf{D}$ and $\mathbf{H}$, we find constitutive equations for the rotating medium, observed in the laboratory frame:

$$\mathbf{D} = \frac{1}{(1-v^2)}\left\{\left(\epsilon - \frac{v^2}{\mu}\right)\mathbf{E} + \left(\epsilon - \frac{1}{\mu}\right)[\mathbf{v}\times\mathbf{B} - (\mathbf{v}\cdot\mathbf{E})\mathbf{v}]\right\}, \tag{26a}$$

$$\mathbf{H} = \frac{1}{(1-v^2)}\left\{\left(\frac{1}{\mu} - \epsilon v^2\right)\mathbf{B} + \left(\epsilon - \frac{1}{\mu}\right)\times[\mathbf{v}\times\mathbf{E} + (\mathbf{v}\cdot\mathbf{B})\mathbf{v}]\right\}. \tag{26b}$$

Upon rearranging Eqs. (26), we find that the resulting equations are identical to Minkowski's constitutive equations, first obtained in 1908 by using special relativity for the case of uniform motion:[6,9–11]

$$\mathbf{D}=\frac{1}{(1-\epsilon\mu\nu^2)}\{\epsilon\mathbf{E}(1-\nu^2)+(\epsilon\mu-1)\times[\mathbf{v}\times\mathbf{H}-\epsilon(\mathbf{v}\cdot\mathbf{E})\mathbf{v}]\}, \tag{27a}$$

$$\mathbf{B}=\frac{1}{(1-\epsilon\mu\nu^2)}\{\mu\mathbf{H}(1-\nu^2)-(\epsilon\mu-1)\times[\mathbf{v}\times\mathbf{E}-\mu(\mathbf{v}\cdot\mathbf{H})\mathbf{v}]\}. \tag{27b}$$

Thus, by starting with Eqs. (1) and using the rules of general covariance, we have been led to Eqs. (27), the special relativity result. This implies that observers in the laboratory reference frame have a choice: they can either start their analysis with Eqs. (1) in rotating coordinates or they can simply use Eqs. (27), obtained on the basis of special relativity.

Another method by which lab-frame observers can derive constitutive equations for a rotating body is simply to use Eqs. (10). Lab-frame observers use the metric for their own flat space–time and the four-velocity of the rotating material as observed in the lab frame. Thus lab-frame observers use

$$\eta_{\alpha\beta}=\mathrm{diag}(1,-1,-1,-1), \tag{28}$$

$$u^{\mu}=\gamma(1,\mathbf{v}), \tag{29}$$

where $\mathbf{v}$ is not uniform, but rather is the rotational velocity of the material with components $\nu_x=-\omega y$ and $\nu_y=\omega x$. Using these equations in Eqs. (10) leads directly to

$$\mathbf{D}+\mathbf{v}\times\mathbf{H}=\epsilon(\mathbf{E}+\mathbf{v}\times\mathbf{B}), \tag{30a}$$

$$\mathbf{B}+\mathbf{E}\times\mathbf{v}=\mu(\mathbf{H}+\mathbf{D}\times\mathbf{v}). \tag{30b}$$

Solving these equations for $\mathbf{D}$ and $\mathbf{B}$ leads directly to Minkowski's constitutive equations,[6,9–11] Eqs. (27). Thus, by using two different methods, we find that the end result agrees with Eqs. (12), derived on the basis of special relativity. This implies that special relativity may be used for rotational motion.

V. DERIVATION OF THE WILSONS' EXPERIMENTAL RESULT

In 1913 M. Wilson and H. A. Wilson performed an experiment in which a cylinder with electric permittivity ϵ and magnetic permeability μ was rotated within a uniform, axially directed magnetic field.[1,11,12] Figure 1 shows a sectional view of a general embodiment of the Wilsons' experiment. The Wilsons measured an electric potential, between inner and outer surfaces of the cylinder, which closely agreed with the special relativistic prediction[1,9,11]

$$\Delta V=\frac{1}{2}\left(\mu-\frac{1}{\epsilon}\right)(r_2^2-r_1^2)\omega\beta_0, \tag{31}$$

where terms of higher order than ν have been neglected. Herein we derive Eq. (31) to demonstrate one method by which special relativity can be used to analyze experiments involving rotational motion.

The Wilsons designed their experiment to preclude free charges and currents; in addition, static fields were used.[1,9,11,12] Under these conditions, observers in the laboratory frame write Maxwell's field equations as

$$\nabla\cdot\mathbf{D}=0, \tag{32a}$$

$$\nabla\cdot\mathbf{B}=0, \tag{32b}$$

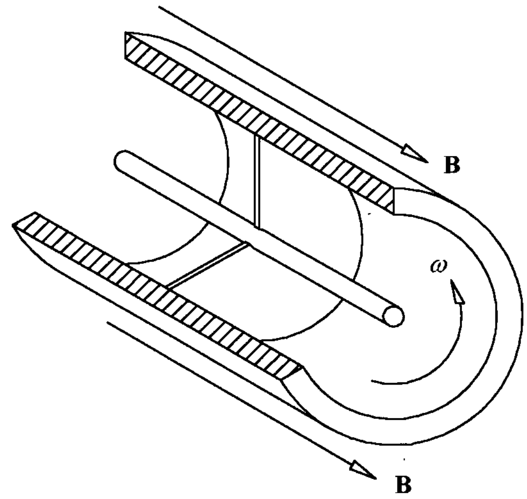

Fig. 1. Sectional view of a general embodiment of the Wilsons' experiment. A long cylinder of polarizable, permeable material rotates in an external, axially directed magnetic field. An electric potential is measured between inner and outer surfaces of the cylinder.

$$\nabla\times\mathbf{E}=0, \tag{32c}$$

$$\nabla\times\mathbf{H}=0. \tag{32d}$$

In addition to these field equations, lab-frame observers use Eqs. (26) and thus rewrite the field equations in the form

$$\nabla\cdot\left\{\frac{1}{(1-\nu^2)}\left[\left(\epsilon-\frac{\nu^2}{\mu}\right)\mathbf{E}+\left(\epsilon-\frac{1}{\mu}\right)[\mathbf{v}\times\mathbf{B}-(\mathbf{v}\cdot\mathbf{E})\mathbf{v}]\right]\right\}=0, \tag{33a}$$

$$\nabla\cdot\mathbf{B}=0, \tag{33b}$$

$$\nabla\times\mathbf{E}=0, \tag{33c}$$

$$\nabla\times\left\{\frac{1}{(1-\nu^2)}\left[\left(\frac{1}{\mu}-\epsilon\nu^2\right)\mathbf{B}+\left(\epsilon-\frac{1}{\mu}\right)\times[\mathbf{v}\times\mathbf{E}+(\mathbf{v}\cdot\mathbf{B})\mathbf{v}]\right]\right\}=0, \tag{33d}$$

where $\mathbf{v}=\omega r\hat{\mathbf{e}}_\phi$ is the rotational velocity of the cylinder. Since the magnetic field is parallel to the axis of the rotating cylinder, and we assume that the cylinder is sufficiently long for end effects to be ignored, Eq. (33a) implies that

$$\left(\epsilon_{\mathrm{IN}}-\frac{\nu^2}{\mu_{\mathrm{IN}}}\right)\mathbf{E}_{\mathrm{IN}}+\left(\epsilon_{\mathrm{IN}}-\frac{1}{\mu_{\mathrm{IN}}}\right)[\mathbf{v}\times\mathbf{B}_{\mathrm{IN}}-(\mathbf{v}\cdot\mathbf{E}_{\mathrm{IN}})\mathbf{v}]=(1-\nu^2)\mathbf{E}_{\mathrm{OUT}}, \tag{34}$$

where the subscripts denote quantities taken inside or outside the material, and we have taken ϵ_{OUT} and μ_{OUT} equal to unity. Outside the material, the only fields present are

$$\mathbf{E}_{\mathrm{OUT}}=0, \tag{35}$$

$$\mathbf{B}_{\mathrm{OUT}}=\mathbf{B}. \tag{36}$$

Using these fields and rearranging somewhat, Eq. (34) becomes

$$(\epsilon\mu-\nu^2)\mathbf{E}_{\mathrm{IN}}=(1-\epsilon\mu)\mathbf{v}\times\mathbf{B}_{\mathrm{IN}}+(\epsilon\mu-1)(\mathbf{v}\cdot\mathbf{E}_{\mathrm{IN}})\mathbf{v}, \tag{37}$$

where the subscripts have been dropped from ϵ and μ. Once again taking into consideration that the magnetic field is parallel to the axis of the rotating cylinder, after some rearranging Eq. (33d) implies that

$$(1-\epsilon\mu)\mathbf{v}\times\mathbf{B}_{\rm IN}=\frac{\mu(1-\epsilon\mu)(1-\nu^2)}{(1-\epsilon\mu\nu^2)}\,\mathbf{v}\times\mathbf{B}+\frac{(\epsilon\mu-1)^2}{(1-\epsilon\mu\nu^2)}\,\mathbf{v}\times(\mathbf{v}\times\mathbf{E}_{\rm IN}). \tag{38}$$

Using Eq. (38) in Eq. (37), and then solving for the electric field inside the material, we find that Eq. (37) becomes

$$\mathbf{E}_{\rm IN}+\frac{(1-\epsilon\mu)}{(1-\nu^2)}\,(\mathbf{v}\cdot\mathbf{E}_{\rm IN})\mathbf{v}=\frac{1}{(1-\nu^2)}\left(\frac{1}{\epsilon}-\mu\right)\mathbf{v}\times\mathbf{B}. \tag{39}$$

To simplify Eq. (39), we take the rotational velocity to be $\mathbf{v}=\omega r\hat{\mathbf{e}}_\phi$, where r is the radius of the cylinder and ω is the angular velocity of rotation. In addition, we take the magnetic field to be directed in the positive z direction: $\mathbf{B}=B_0\hat{\mathbf{e}}_z$. Taking note that in the case of a cylinder rotating about its axis, the electric field $\mathbf{E}_{\rm IN}$ is directed perpendicular to both the velocity $\mathbf{v}$ and the magnetic field $\mathbf{B}$,[22] we find that Eq. (39) simplifies to

$$\mathbf{E}_{\rm IN}=\frac{\omega r}{(1-\nu^2)}\left(\frac{1}{\epsilon}-\mu\right)B_0\hat{\mathbf{e}}_r. \tag{40}$$

When the rotational velocity is taken to be very much smaller than the velocity of light, terms of order higher than ν may be neglected. Upon carrying this out, we find that the electric field within the material of the cylinder simplifies to

$$\mathbf{E}_{\rm IN}=\omega r\left(\frac{1}{\epsilon}-\mu\right)B_0\hat{\mathbf{e}}_r. \tag{41}$$

This radially directed electric field leads to the electric potential of Eq. (31); thus we have successfully used special relativity, in conjunction with Maxwell's equations, to derive the Wilsons' experimental result[1,9,11] in the lab frame.

VI. POLARIZATION AND MAGNETIZATION

As shown in Sec. IV, constitutive equations in the rotating frame can be transformed to the laboratory frame by use of Eqs. (24) and Eqs. (25). Herein, we use a similar approach to find the polarization and magnetization observed in the lab frame, but expressed in terms of fields observed in the rotating frame. We begin by noting that the polarization $\mathbf{P}'$ and magnetization $\mathbf{M}'$ in the rotating frame can be related to the polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ in the lab frame by a transformation of the form[9,10,21]

$$\mathbf{P}'=\mathbf{P}-\mathbf{v}\times\mathbf{M}, \tag{42a}$$

$$\mathbf{M}'=\mathbf{M}, \tag{42b}$$

where quantities in the rotating frame carry a prime, and we have neglected terms of order higher than ν. An inverse transformation can be obtained by simply switching primed and unprimed quantities, and reversing the sign of the velocity:

$$\mathbf{P}=\mathbf{P}'+\mathbf{v}\times\mathbf{M}', \tag{43a}$$

$$\mathbf{M}=\mathbf{M}'. \tag{43b}$$

When written in terms of Eqs. (23), the polarization and magnetization observed in the lab frame become

$$\mathbf{P}=\frac{1}{4\pi}\,(\epsilon-1)\mathbf{E}'+\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\mathbf{v}\times\mathbf{B}', \tag{44a}$$

$$\mathbf{M}=\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\mathbf{B}'+\frac{1}{4\pi}\left(\frac{1}{\mu}-\epsilon\right)\mathbf{v}\times\mathbf{E}'. \tag{44b}$$

With a knowledge of the fields observed in the rotating frame, Eqs. (44) can be used to determine the polarization and magnetization observed in the lab frame.

We first consider the case in which a polarizable cylinder rotates in the lab frame, in an external electric field directed along the axis of rotation: $\mathbf{E}\neq\mathbf{0}$ and $\mathbf{B}=\mathbf{0}$. According to Eqs. (24), the fields observed in the rotating frame are then $\mathbf{E}'=\mathbf{E}$ and $\mathbf{B}'=\mathbf{0}$. Using these fields in Eqs. (44), and noting that for a nonmagnetizable cylinder $\mu=1$, we find that the polarization and magnetization observed in the lab frame are

$$\mathbf{P}=\frac{1}{4\pi}\,(\epsilon-1)\mathbf{E}', \tag{45a}$$

$$\mathbf{M}=\frac{1}{4\pi}\,(1-\epsilon)\mathbf{v}\times\mathbf{E}'. \tag{45b}$$

Thus, according to Eq. (45b), a rotating polarized cylinder gives rise to a magnetization in the lab frame. Upon comparing Eqs. (45) with Eqs. (23), we find that observers in the lab frame detect a polarization and magnetization given by

$$\mathbf{P}=\mathbf{P}', \tag{46a}$$

$$\mathbf{M}=\mathbf{P}'\times\mathbf{v}, \tag{46b}$$

where quantities in the rotating frame carry a prime, and $\mathbf{v}=\omega r\hat{\mathbf{e}}_\phi$ is the rotational velocity of the cylinder relative to the lab frame. Equations (46) agree with known experimental results, and can be derived on the basis of special relativity.[10]

Equations (44) can also be used to analyze the case of a magnetizable cylinder rotating in the lab frame, in an external, axially directed magnetic field. Substituting the lab-frame fields $\mathbf{E}=\mathbf{0}$ and $\mathbf{B}\neq\mathbf{0}$ into Eqs. (24), we find that the fields observed in the rotating frame are $\mathbf{E}'=\mathbf{v}\times\mathbf{B}$ and $\mathbf{B}'=\mathbf{B}$. Using these fields in Eqs. (44), and noting that $\epsilon=1$ for a nonpolarizable cylinder, we find that the polarization and magnetization observed in the lab frame are

$$\mathbf{P}=\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\mathbf{v}\times\mathbf{B}', \tag{47a}$$

$$\mathbf{M}=\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\mathbf{B}'. \tag{47b}$$

When Eqs. (47) are compared to Eqs. (23), we find that lab-frame observers detect a polarization and magnetization according to

$$\mathbf{P}=\mathbf{v}\times\mathbf{M}', \tag{48a}$$

$$\mathbf{M}=\mathbf{M}', \tag{48b}$$

where as before, quantities in the rotating frame carry a prime, and $\mathbf{v}=\omega r\hat{\mathbf{e}}_\phi$ is the rotational velocity of the cylinder relative to the lab frame. Just as in the case of Eqs. (46), Eqs. (48) agree with known experimental results, and can be derived on the basis of special relativity.[10]

VII. CONCLUSIONS

We have shown one method by which relativistic electrodynamics can be extended to include rotating linear media. We first derived general field equations for the rotating reference frame, expressed in three-dimensional notation. Fields in the presence of matter were then related to the electric and magnetic fields by use of Minkowski's covariant constitutive equations.[6,8–10] We showed that using Minkowski's covariant constitutive equations in rotating coordinates leads directly to the correct constitutive equations and hence to the correct field equations in the rotating frame.[1] In addition, we showed that Minkowski's covariant constitutive equations can also be used to derive the polarization and magnetization in the rotating frame.[9,21]

Two methods were used to derive the constitutive equations in the laboratory reference frame. In the first method, field transformations were used to transform the constitutive equations from the rotating frame to the laboratory frame; in the second method, Minkowski's covariant constitutive equations[6,8–10] were used directly in the laboratory frame with the four-velocity of the rotating medium. Upon rearranging the lab-frame constitutive equations, we found agreement with the 1908 constitutive equations of Minkowski,[6,9–11] obtained on the basis of special relativity. Thus it is readily apparent that special relativity can be used to analyze experiments involving rotational motion. To exemplify the use of special relativity, we used the laboratory constitutive equations in conjunction with Maxwell's field equations to derive the electric potential observed in the 1913 experiment of Wilson and Wilson.[1,9,11]

As a further demonstration of the special relativity connection, transformations were used to relate the polarization and magnetization in the laboratory frame to fields observed in the rotating frame. We then considered the case in which a polarizable cylinder rotates in the lab frame, in an external, axially directed electric field. Upon using the fields observed in the rotating frame in the transformation equations, and setting the magnetic permeability equal to unity, we found that lab-frame observers detect a polarization and magnetization in agreement with the predictions of special relativity.[10] Similarly, we considered the case in which a magnetizable cylinder rotates in the lab frame, in an external, axially directed magnetic field. As before, we used the fields observed in the rotating frame in the transformation equations, but this time we set the electric permittivity equal to unity. Upon carrying this out, we found once again that lab-frame observers detect a polarization and magnetization in accordance with the predictions of special relativity.[10]

We conclude that relativistic electrodynamics can be used for rotating linear media only when one uses a covariant form of the constitutive equations. More particularly, we conclude that special relativity can indeed be used in the analysis of experiments involving rotational motion.

APPENDIX

We wish to derive a transformation that relates fields observed in the rotating reference frame to those observed in the laboratory frame. We begin by noting that the metric tensor in the rotating coordinate system is[1,3,23]

$$g'_{\alpha\beta}=\begin{pmatrix} 1-v^2 & -v_x & -v_y & 0 \\ -v_x & -1 & 0 & 0 \\ -v_y & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A1}$$

where we have taken $v_x=-\omega y'$, $v_y=\omega x'$, and $v^2=v_x^2+v_y^2$. The inverse metric tensor in rotating coordinates is[3]

$$g'^{\alpha\beta}=\begin{pmatrix} 1 & -v_x & -v_y & 0 \\ -v_x & -1+v_x^2 & v_x v_y & 0 \\ -v_y & v_x v_y & -1+v_y^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A2}$$

Using the metric tensor in rotating coordinates and the transformation for a covariant second rank tensor,[2]

$$\eta''_{\alpha\beta}=g'_{\mu\nu}\,\frac{\partial x'^{\mu}}{\partial x''^{\alpha}}\,\frac{\partial x'^{\nu}}{\partial x''^{\beta}}, \tag{A3}$$

we obtain a transformation which relates quantities in the rotating frame to those in a momentarily comoving reference frame (MCRF) of an observer at rest in the rotating frame:[20]

$$L^{\alpha}_{\beta}=\frac{\partial x'^{\alpha}}{\partial x''^{\beta}}=\begin{pmatrix} \gamma & v_x\gamma^2 & v_y\gamma^2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A4}$$

where $\gamma=1/\sqrt{1-v^2}$. We can now use Eq. (A4) to transform the electromagnetic tensor from the rotating frame to the observer's MCRF by using

$$F''=L^T F' L, \tag{A5}$$

where F'' is the electromagnetic tensor in the MCRF, and L^T is the transpose of Eq. (A4). Upon carrying out the matrix multiplication in Eq. (A5), we find that fields in the rotating frame are related to those in the MCRF by

$$\mathbf{E}''=\gamma\mathbf{E}', \tag{A6a}$$

$$\mathbf{B}''=\mathbf{B}'-\gamma^2(\mathbf{v}\times\mathbf{E}'), \tag{A6b}$$

where quantities in the MCRF carry a double prime.

Although the rotating frame has a rotational velocity $\mathbf{v}=\omega r\hat{\mathbf{e}}_\phi$ relative to the lab frame, at any given instant, the MCRF appears to have a uniform velocity $\mathbf{v}$ relative to the lab frame. Thus, with Eqs. (A6) in hand, we can relate fields in the MCRF to those observed in the lab frame by using the well-known special relativity transformation[13]

$$\mathbf{E}''=\gamma(\mathbf{E}+\mathbf{v}\times\mathbf{B})-\frac{\gamma^2}{\gamma+1}\,(\mathbf{v}\cdot\mathbf{E})\mathbf{v}, \tag{A7a}$$

$$\mathbf{B}''=\gamma(\mathbf{B}-\mathbf{v}\times\mathbf{E})-\frac{\gamma^2}{\gamma+1}\,(\mathbf{v}\cdot\mathbf{B})\mathbf{v}. \tag{A7b}$$

By equating Eqs. (A6) and Eqs. (A7) and solving for quantities in the rotating frame, we obtain a transformation that relates fields in the rotating frame to fields in the laboratory frame:

$$\mathbf{E}'=\mathbf{E}+\mathbf{v}\times\mathbf{B}-\frac{\gamma}{\gamma+1}\,(\mathbf{v}\cdot\mathbf{E})\mathbf{v}, \tag{A8a}$$

$$\mathbf{B}'=\gamma(\mathbf{B}-\mathbf{v}\times\mathbf{E})+\gamma^2\mathbf{v}\times(\mathbf{E}+\mathbf{v}\times\mathbf{B})-\frac{\gamma^2}{\gamma+1}\,(\mathbf{v}\cdot\mathbf{B})\mathbf{v}. \tag{A8b}$$

Using the same procedure used to obtain Eqs. (A8), analogous transformations for auxiliary fields $\mathbf{D}'$ and $\mathbf{H}'$ are also obtained:

$$\mathbf{D}'=\mathbf{D}+\mathbf{v}\times\mathbf{H}-\frac{\gamma}{\gamma+1}\,(\mathbf{v}\cdot\mathbf{D})\mathbf{v}, \tag{A9a}$$

$$\mathbf{H}' = \gamma(\mathbf{H} - \mathbf{v}\times\mathbf{D}) + \gamma^2\mathbf{v}\times(\mathbf{D}+\mathbf{v}\times\mathbf{H}) - \frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\mathbf{H})\mathbf{v}. \tag{A9b}$$

We note that when the speed of rotation is taken to be very much less than the speed of light, Eqs. (A8) simplify to a more familiar form:[4,24]

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B}, \tag{A10a}$$

$$\mathbf{B}' = \mathbf{B}. \tag{A10b}$$

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[17]Although we have chosen to define the electric and magnetic fields as components of a covariant tensor $F_{\alpha\beta}$, we could also have chosen the fields to be components of a contravariant tensor $F^{\alpha\beta}$. So long as one is consistent throughout the analysis, either the covariant or contravariant convention may be used.
[18]Here, we have referred to **D** and **H** as the electric and magnetic fields in matter; however, **D** and **H** are actually well defined both inside and outside of matter, as are the true electric and magnetic fields, **E** and **B**. Since the fields **D** and **H** are simply useful in calculating **E** and **B** when matter is present, to avoid confusion, we will refer to **E** and **B** as fundamental fields, and to **D** and **H** as auxiliary fields.
[19]The covariant derivative of the electromagnetic tensor may be written as $\boldsymbol{\nabla}_\alpha F^{\alpha\beta} = (-g)^{-1/2}\partial_\alpha(\sqrt{-g}F^{\alpha\beta})$, where g is the determinant of the metric tensor. In rotating coordinates $\sqrt{-g}=1$; and as a result, the covariant derivative is exactly equal to the ordinary partial derivative.
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