Vector-based, quick & dirty response to the issues posed by Pellegrini

August 16, 2008

In my previous write-up, fields were expressed exclusively in terms of a 1-form, co-vector basis. I demonstrated how these fields could be shown to be consistent with the fundamental field equations when material sources are included. The analysis led to expressions for the material electric dipole moment density and magnetic dipole moment density that were physically confusing, even suggesting the possibility of a new physical phenomenon in the rotating coordinate system. To clear up any confusion on the matter, the following vector-based analysis is offered. In the following, all fields are vectors (i.e., components of second rank contravariant tensors) expressed in the rotating reference frame. As usual, we restrict things to first order in the velocity and set the speed of light equal to unity.

Since we are working in the vector basis, we may begin with vector field equations, given in my 1999 paper as Eqs. (10):

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f \tag{1a}$$

$$\vec{\nabla} \times \vec{H} = 4\pi \vec{j}_f \tag{1b}$$

Using the contravariant electromagnetic tensor, given by Eq. (7) in the 1999 paper, in Eq. (4) of the 1998 paper leads to

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_T \tag{2a}$$

$$\vec{\nabla} \times \vec{B} = 4\pi \vec{j}_T \tag{2b}$$

According to Pellegrini, we should be able to add the bound charge and current densities to the above Eqs. (1), respectively, and then arrive at the above Eqs. (2). Pellegrini suggests that we carry this out by using

$$\vec{\nabla} \cdot \vec{P} = -\rho_b \tag{3a}$$

$$\vec{\nabla} \times \vec{M} = \vec{j}_b \tag{3b}$$

In my last write-up, I insisted that we cannot merely assume that these equations hold in any frame. So, with that in mind, I shall begin by deriving the above Eqs. (3) in terms of vectors in the rotating frame. As pointed out in my 1999 paper, we must choose vectors or 1-forms and then stick with our decision throughout the analysis. This is a vector-based analysis, thus I begin with the vector polarization is given by Eq. (25a) in the 1999 paper:

$$\vec{P} = \frac{1}{4\pi}(\varepsilon - 1)\vec{E} + \frac{1}{4\pi}\left(\varepsilon - \frac{1}{\mu}\right)\vec{v} \times \vec{B} \tag{4}$$

Taking the divergence of the polarization gives

$$\vec{\nabla}\cdot\vec{P}=\frac{1}{4\pi}(\varepsilon-1)\vec{\nabla}\cdot\vec{E}+\frac{1}{4\pi}\left(\varepsilon-\frac{1}{\mu}\right)\vec{\nabla}\cdot\left(\vec{v}\times\vec{B}\right) \tag{5}$$

This can be simplified by using Eq. (16a):

$$\vec{D}=\varepsilon\vec{E}+\left(\varepsilon-\frac{1}{\mu}\right)\vec{v}\times\vec{B} \tag{6}$$

Taking the divergence and rearranging a bit leads to

$$\varepsilon\vec{\nabla}\cdot\vec{E}=\vec{\nabla}\cdot\vec{D}-\left(\varepsilon-\frac{1}{\mu}\right)\vec{\nabla}\cdot\left(\vec{v}\times\vec{B}\right) \tag{7}$$

Direct substitution of this expression into Eq. (5) above gives

$$\vec{\nabla}\cdot\vec{P}=\frac{1}{4\pi}\left[\vec{\nabla}\cdot\vec{D}-\left(\varepsilon-\frac{1}{\mu}\right)\vec{\nabla}\cdot\left(\vec{v}\times\vec{B}\right)-\vec{\nabla}\cdot\vec{E}\right]+\frac{1}{4\pi}\left(\varepsilon-\frac{1}{\mu}\right)\vec{\nabla}\cdot\left(\vec{v}\times\vec{B}\right) \tag{8}$$

which leads directly to

$$\vec{\nabla}\cdot\vec{P}=\frac{1}{4\pi}\left(\vec{\nabla}\cdot\vec{D}-\vec{\nabla}\cdot\vec{E}\right) \tag{9}$$

Upon using the above Eqs. (1a) and (2a), we find that

$$\vec{\nabla}\cdot\vec{P}=-\rho_b$$

This equation is identical to Eq. (3a) above, as proposed by Pellegrini, but it only comes about when vectors are chosen from the beginning. As a reminder, had we used 1-forms throughout the analysis, we would have arrived at an expression of the form:

$$\vec{\nabla}\cdot\tilde{P}=-\rho_b+\frac{1}{4\pi}\left(\frac{1}{\mu}-1\right)\vec{\nabla}\cdot\left(\vec{v}\times\tilde{B}\right)$$

where $\tilde{P}$ and $\tilde{B}$ are defined with respect to a 1-form, co-vector basis, as demonstrated in my 1999 paper. The second term on the right-hand side of this equation is confusing only until we recognize that it is expressed in terms of co-vectors, have a 1-form basis.

With a trusted version of Eq. (3a) in hand, we can now perform Pellegrini's check for consistency. We begin with the *vector* field equation given as Eq. (17a) in the 1999 paper, and add the bound charge density to both sides:

$$\vec{\nabla}\cdot\left\{\varepsilon\vec{E}+\left(\varepsilon-\frac{1}{\mu}\right)\vec{v}\times\vec{B}\right\}+4\pi\rho_b=4\pi\rho_f+4\pi\rho_b \tag{10}$$

Using Eq. (3a) and remembering that $\rho_T=\rho_f+\rho_b$, we arrive at

$$\vec{\nabla}\cdot\left\{\varepsilon\vec{E}+\left(\varepsilon-\frac{1}{\mu}\right)\vec{v}\times\vec{B}-4\pi\vec{P}\right\}=4\pi\rho_T$$

Using Eq. (4) above leads directly to

$$\vec{\nabla}\cdot\vec{E}=4\pi\rho_T$$

As can be seen, this expression is identical to the above Eq. (2a). It is clear that Eq. (1a) above passes Pellegrini's check for consistency.

Turning our attention now to the above Eq. (1b), we may begin with the vector magnetization, given as Eq. (25b) in my 1999 paper:

$$\vec{M}=\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\vec{B} \tag{11}$$

Taking the curl of Eq. (11) gives

$$\vec{\nabla}\times\vec{M}=\frac{1}{4\pi}\left(1-\frac{1}{\mu}\right)\vec{\nabla}\times\vec{B} \tag{12}$$

This can be simplified by using vector Eq. (16b) of the 1999 paper:

$$\vec{H}=\frac{1}{\mu}\vec{B} \tag{13}$$

Taking the curl of both sides of Eq. (13) and substituting into Eq. (12) leads to

$$\vec{\nabla}\times\vec{M}=\frac{1}{4\pi}\left(\vec{\nabla}\times\vec{B}-\vec{\nabla}\times\vec{H}\right) \tag{14}$$

Upon referring to Eqs. (1b) and (2b), it is straightforward to see that Eq. (14) simplifies to

$$\vec{\nabla}\times\vec{M}=\vec{j}_b$$

Again, we arrive at an expression identical with Pellegrini's equations, given above as Eq. (3b), but only for the case when *vectors* are used throughout the analysis. Consider, had we used 1-forms, Eq. (3b) would have taken on the form:

$$\vec{\nabla}\times\widetilde{M}=\vec{j}_b-\frac{1}{4\pi}(\varepsilon-1)\vec{\nabla}\times\left(\vec{v}\times\widetilde{E}\right)$$

where $\widetilde{P}$ and $\widetilde{E}$ are defined with respect to a 1-form, co-vector basis.

According to Pellegrini, adding the bound current density to both sides of Eq. (17d) in my 1999 paper ought to reproduce Eq. (2b) if all is consistent:

$$\vec{\nabla}\times\frac{1}{\mu}\vec{B}+4\pi\vec{j}_b=4\pi\vec{j}_f+4\pi\vec{j}_b \tag{15}$$

Using Pellegrini's Eq. (3b) in Eq. (15) gives

$$\vec{\nabla}\times\left(\frac{1}{\mu}\vec{B}+4\pi\vec{M}\right)=4\pi\vec{j}_T \tag{16}$$

Upon substituting the vector magnetization, given above as Eq. (11), into Eq. (16) and simplifying a bit, we arrive at

$$\vec{\nabla}\times\vec{B}=4\pi\vec{j}_T$$

Again, we have obtained an equation identical to Eq. (2b). Thus, Eq. (1b) passes Pellegrini's check for consistency.

When the above is coupled with my previous write-up, wherein 1-form fields were used, it appears that all the field equations given in my 1999 paper fulfill Pellegrini's check for consistency. Based on this, I assert that the confusion surrounding rotating frames is hinged entirely on the choice of covariant versus contravariant second rank tensors. In an inertial frame, both vectors and 1-forms lead to equations of the same form. This comes about only because the metric tensor has a *nice*, unitary form in inertial frames. But the instant the metric tensor starts acting up and gets more complicated, we must be careful with our beginning definitions, and then we must stick with the definitions we have chosen. Notice in my 1999 paper, how vector and co-vector equations in the rotating frame, such as Eq. (37) and (42), have a different form, but in the inertial lab frame they have the same form, as evidenced by Eqs. (38) and (43). This is just one example of how vectors and 1-forms differ in the rotating frame, but merry up in the lab frame. ■