Quick & Dirty response to the issues posed by Pellegrini

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A couple of issues come to mind. One issue surrounds our choice of second rank tensors. As pointed out in my 1999 paper, and also mentioned by Kirk McDonald and Schiff, we may choose to work with covariant, contravariant, or mixed tensors, but we must stick with whatever choice we make. In the rotating frame, things don't work like they do in an inertial frame. We cannot naively define the covariant F and the contravariant M and then throw them into the covariant field equation without obtaining spurious results. Another issue is Pellegrini's assumption that the expressions

$$\nabla \cdot \vec{P} = -\rho_b$$

$$\nabla \times \vec{M} = \vec{j}_b$$

hold generally in any coordinate system; I don't think they do. In the following, I perform the "check" that Pellegrini mentions, but I do not assume the above expressions; rather, I derive expressions for the rotating frame. I use the resources developed in my 1999 paper, so even though I herein write fields as vectors, it should be noted these fields are components of the covariant second rank tensors F, H and M, and thus are "one-forms," also known as "co-vectors." All entities are in the rotating frame and the speed of light is set to unity.

In the rotating frame, we have my Eq. (6a) from my 1999 paper:

$$\vec{\nabla} \cdot (\vec{D} - \vec{v} \times \vec{H}) = 4\pi\rho_f \qquad \text{Eq. (1a)}$$

According to Eq. (4) of my 1998 paper, we also have:

$$\vec{\nabla} \cdot (\vec{E} - \vec{v} \times \vec{B}) = 4\pi\rho_T \qquad \text{Eq. (1b)}$$

In the 1999 paper, I show that the polarization in the rotating frame is:

$$\vec{P} = \frac{1}{4\pi}(\varepsilon - 1)\vec{E} \qquad \text{Eq. (1c)}$$

The divergence of the polarization may be put in the form:

$$\nabla \cdot \vec{P} = \frac{1}{4\pi}\left(\nabla \cdot \vec{D} - \nabla \cdot \vec{E}\right) \qquad \text{Eq. (2)}$$

Rearranging Eqs. (1a) and (1b) gives:

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f + \vec{\nabla} \cdot (\vec{v} \times \vec{H}) \qquad \text{Eq. (3a)}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_T + \vec{\nabla} \cdot (\vec{v} \times \vec{B}) \qquad \text{Eq. (3b)}$$

In the 1999 paper, we also have Eq. (13b):

$$\vec{H} = \frac{1}{\mu}\vec{B} + \frac{1}{(1 - v^2)}\left(\varepsilon - \frac{1}{\mu}\right)\vec{v} \times \vec{E}$$

And upon keeping only first order terms, we have

$$\vec{v} \times \vec{H} = \frac{1}{\mu} \vec{v} \times \vec{B}$$

Using this relation in Eq. (3a) leads directly to

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f + \frac{1}{\mu} \vec{\nabla} \cdot (\vec{v} \times \vec{B}) \qquad \text{Eq. (4)}$$

Substituting Eqs. (3b) and (4) into Eq. (2) gives

$$\nabla \cdot \vec{P} = \frac{1}{4\pi}\left(4\pi\rho_f - 4\pi\rho_T + \frac{1}{\mu}\vec{\nabla} \cdot \left(\vec{v} \times \vec{B}\right) - \vec{\nabla} \cdot (\vec{v} \times \vec{B})\right)$$

Upon simplifying a bit and using $\rho_T = \rho_f + \rho_b$, we arrive at an expression for the divergence of the polarization in the rotating frame:

$$\nabla \cdot \vec{P} = -\rho_b + \frac{1}{4\pi}\left(\frac{1}{\mu} - 1\right) \vec{\nabla} \cdot (\vec{v} \times \vec{B}) \qquad \text{Eq. (5)}$$

Equation (5) differs from the usual inertial expression by the presence of an additional term on the right-hand side. Now we may return to the issue Pellegrini has raised regarding the validity of Eqs. (8) in my 1998 paper. Keeping with the spirit of the above analysis, I shall use Eq. (14a) of my 1999 paper, taken to first order in the velocity. According to Pellegrini, we ought to be able to add the bound charge density to both sides of Eq. (14a) of my 1999 paper and ultimately arrive at Eq. (1b) above. Carrying this out, we begin with

$$\vec{\nabla} \cdot \left(\varepsilon\vec{E} - \vec{v} \times \frac{1}{\mu}\vec{B}\right) + 4\pi\rho_b = 4\pi\rho_f + 4\pi\rho_b \qquad \text{Eq. (6)}$$

Upon rearranging Eq. (5), we have

$$4\pi\rho_b = -4\pi\nabla \cdot \vec{P} + \left(\frac{1}{\mu} - 1\right) \vec{\nabla} \cdot (\vec{v} \times \vec{B})$$

which when substituted into Eq. (6) gives

$$\vec{\nabla} \cdot \left(\varepsilon\vec{E} - \vec{v} \times \frac{1}{\mu}\vec{B} - 4\pi\vec{P} + \left(\frac{1}{\mu} - 1\right)(\vec{v} \times \vec{B})\right) = 4\pi\rho_T \qquad \text{Eq. (7)}$$

Returning again to Eq. (1c), it is easy to see that we may put

$$-4\pi\vec{P} = \vec{E} - \varepsilon\vec{E}$$

Using this in Eq. (7) leaves us with an expression identical with Eq. (1b):

$$\vec{\nabla} \cdot \left(\vec{E} - \vec{v} \times \vec{B}\right) = 4\pi\rho_T$$

This is the first expression sought by Pellegrini.

Next, we turn to the issue of bound currents. As mentioned above, rather than assuming the expression

$$\nabla \times \vec{M} = \vec{J}_b$$

holds in all coordinate systems, I shall derive an expression for the rotating frame. Equation (6d) in my 1999 paper is:

$$\vec{\nabla} \times (\vec{H} - \vec{v} \times \vec{D}) = 4\pi \vec{J}_f \qquad \text{Eq. (8a)}$$

And, according to Eq. (4) of my 1998 paper, we also have:

$$\vec{\nabla} \times (\vec{B} - \vec{v} \times \vec{E}) = 4\pi \vec{J}_T \qquad \text{Eq. (8b)}$$

The magnetization in the rotating frame is given by Eq. (21b):

$$\vec{M} = \frac{1}{4\pi}\left(1 - \frac{1}{\mu}\right)\vec{B} + \frac{1}{4\pi(1 - v^2)}\left(\frac{1}{\mu} - \varepsilon\right)\vec{v} \times \vec{E}$$

Equations (13) are:

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{H} = \frac{1}{\mu}\vec{B} + \frac{1}{(1 - v^2)}\left(\varepsilon - \frac{1}{\mu}\right)\vec{v} \times \vec{E}$$

Taking the curl of the magnetization gives:

$$\nabla \times \vec{M} = \frac{1}{4\pi}\left(1 - \frac{1}{\mu}\right)\nabla \times \vec{B} + \frac{1}{4\pi(1-v^2)}\left(\frac{1}{\mu} - \varepsilon\right)\nabla \times \left(\vec{v} \times \vec{E}\right) \qquad \text{Eq. (9)}$$

Taking the curl of H gives:

$$\nabla \times \vec{H} = \frac{1}{\mu}\nabla \times \vec{B} + \frac{1}{(1-v^2)}\left(\varepsilon - \frac{1}{\mu}\right)\nabla \times \left(\vec{v} \times \vec{E}\right) \qquad \text{Eq. (10)}$$

Substituting Eq. (10) into Eq. (9), and simplifying a bit, leads to

$$\nabla \times \vec{M} = \frac{1}{4\pi}\nabla \times \vec{B} - \frac{1}{4\pi}\nabla \times \vec{H} \qquad \text{Eq. (11)}$$

We may recast Eqs. (8) as:

$$\vec{\nabla} \times \vec{H} = 4\pi \vec{J}_f + \vec{\nabla} \times (\vec{v} \times \vec{D}) \qquad \text{Eq. (12a)}$$

$$\vec{\nabla} \times \vec{B} = 4\pi \vec{J}_T + \vec{\nabla} \times (\vec{v} \times \vec{E}) \qquad \text{Eq. (12b)}$$

Substituting Eqs. (12) into Eq. (11) leads to

$$\nabla \times \vec{M} = \vec{J}_T - \vec{J}_f + \frac{1}{4\pi} \nabla \times \left[(\vec{v} \times \vec{E}) - \varepsilon (\vec{v} \times \vec{E}) \right] \qquad \text{Eq. (13)}$$

where I have used the expression

$$\vec{D} = \varepsilon \vec{E}$$

Upon noting that $\vec{J}_T = \vec{J}_b + \vec{J}_f$ and solving the bound current, we arrive at

$$\vec{J}_b = \nabla \times \vec{M} + \frac{(\varepsilon - 1)}{4\pi} \nabla \times (\vec{v} \times \vec{E}) \qquad \text{Eq. (14)}$$

With Eq. (14) in hand, we can address the issue raised by Pellegrini. According to Pellegrini, we should to be able to add the bound current density to both sides of Eq. (14d) of my 1999 paper and arrive at Eq. (8b) above. We may begin with the expression

$$\vec{\nabla} \times \frac{1}{\mu} \left(\vec{B} - \vec{v} \times \vec{E} \right) + 4\pi \vec{J}_b = 4\pi \vec{J}_f + 4\pi \vec{J}_b \qquad \text{Eq. (15)}$$

$$\vec{\nabla} \times \left[\frac{1}{\mu} \left(\vec{B} - \vec{v} \times \vec{E} \right) + 4\pi \vec{M} + (\varepsilon - 1)(\vec{v} \times \vec{E}) \right] = 4\pi \vec{J}_T \qquad \text{Eq. (16)}$$

This expression can be further simplified by noticing that

$$4\pi \vec{M} = \left(1 - \frac{1}{\mu} \right) \vec{B} + \left(\frac{1}{\mu} - \varepsilon \right) \vec{v} \times \vec{E} \qquad \text{Eq. (17)}$$

Substituting Eq. (17) into Eq. (16) and simplifying leads directly to

$$\vec{\nabla} \times \left(\vec{B} - \vec{v} \times \vec{E} \right) = 4\pi \vec{J}_T \qquad \text{Eq. (18)}$$

It is straightforward to see that this expression is identical to Eq. (8b) above, and fulfills the "check" for consistency proposed by Pellegrini.