On the Nature of Inertia

Charles T. Ridgely
2936 Maple Avenue
Fullerton, California 92835

We show that special and general relativity provide a clear statement regarding the origin of inertia. We begin by pointing out that according to general relativity, inertial and gravitational forces arise directly out of the structure of space-time. This implies that an adequate description of inertia cannot be obtained solely on the basis of inertial mass; the role played by space-time must also be considered. General relativity is then used to derive the proper force experienced by a uniformly accelerating observer. Next, we use special relativity to derive the inertial resistance of an observer accelerating under the influence of a constant external force in flat, Minkowski space-time. Both analyses imply that the origin of inertia resides in a relationship between the energy content of matter and the local structure of time. We ultimately conclude that the origin of inertia is the relativistic nature of time.

1. Introduction

Certainly one of the longest standing questions in the realm of physics is by what means do material bodies resist changes to their states of motion (inertia). Historically there have been two approaches to the problem of inertia. On one hand, inertia has been viewed as a fundamental property of matter capable of no further explanation; and on the other hand, inertia has been viewed as arising out of a gravitational coupling among all matter in the universe. The latter view, first proposed by Mach (ca. 1883), arises out of the notion that relative motion loses its meaning in the absence of surrounding matter. This being the case, one is naturally led to the conclusion that the inertial properties of matter must be somehow asymptotically related to the cosmic distribution of all other matter in the universe. Unfortunately, Mach's principle has not led to any greater understanding of inertia other than that it is a fundamental property of matter.

In more recent studies, a typical approach to the inertia problem has been an attempt to express the inertial mass, m, appearing in Newton’s second law of motion,

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{1}$$

in terms of some other entity or interaction. One noteworthy approach to this end has been put forth by B. Haisch, A. Rueda, and H. E. Puthoff [1-4]. In their analysis, it is proposed that the inertial properties of ordinary matter arise out of an interaction between the quarks and electrons constituting matter and the vacuum electromagnetic zero-point field (ZPF). Although the calculations are laborious at best, the ZPF approach seems appealing because it suggests a local electromagnetic basis for gravitational and inertial forces, while avoiding the need for action at a distance phenomena often associated with Mach's principle [1-4]. However, the ZPF approach is not without conceptual difficulties.

One basic problem is that the quarks and electrons constituting matter must themselves be endowed with an inertial mass of their own. As is well known, subatomic particles can be each associated with a rest-energy, which is each particle's internal energy content in the absence of motion. According to $E = mc^2$, called the "law of inertia of energy" in earlier times [5], the rest-energy of subatomic particles must give rise to inertial effects in addition to those induced by the ZPF. In this sense, if the inertial properties of ordinary matter are shown to arise out of an interaction between subatomic particles and the ZPF; but then inertial properties must also be ascribed to the particles themselves, then the problem of inertia has been merely shifted to a smaller quantity of matter rather than truly solved.

Although we do not herein debate the substantiality of the ZPF hypothesis, we must point out that whether the interaction between matter and the ZPF contributes to the inertia of matter remains an open question. Assuming the ZPF does indeed contribute to inertia, the next natural question to ask is what percentage of inertia is actually ZPF induced. This question seems to be hinged on how one interprets $E = mc^2$. According to the ZPF proposal, the rest mass-energy of matter is actually internal kinetic energy that zero-point fluctuations induce on the quarks and electrons constituting matter [4]. We, on the other hand, choose the more traditional interpretation that $E = mc^2$ is an explicit statement regarding the inertial properties of energy. All forms of energy exhibit the property of inertia according to $m = E / c^2$ [6-7]. It is an experimental fact that any body possessing non-zero mass will resist changes to its state of motion in accordance with Eq. (1) and its relativistic generalization. Taken in conjunction with $E = mc^2$, this statement becomes generalized to include all forms of energy [6-7]. In our opinion, therefore, the origin of inertia must necessarily reside in some entity or interaction that is more fundamental than material inertial mass.

One important point to recognize is that according to general relativity, inertial and gravitational forces alike arise out of the structure of space-time [8-13]. This implies that an adequate description of inertia cannot be obtained solely on the basis of the inertial mass m appearing in Eq. (1); the role played by space-time must also be considered. Thus, whether considering the origin of inertia or gravitation, one must ultimately consider the more general question: By what means does space-time structure act upon matter to create an observable force? Clearly, it is a primary objective of the ZPF theory to provide an answer to this question. It is an objective of this paper, however, to answer this question on the basis of special and general relativity; and in the process, to provide a clear and simple statement regarding the origin of inertia.

2. The Proper Force Experienced by a Uniformly Accelerating Observer

As pointed out in the Introduction, general relativity makes clear that the origin of both inertial and gravitational forces lies in the structure of space-time [8-13]. In this section, we demonstrate this by using general relativity to derive the proper force experienced by an observer accelerating uniformly under the influence of a constant external force.

We begin by noting that so long as the external force remains constant, so too does the acceleration experienced by the observer. As a result, the accelerating observer's local system of coordinates remains fixed with respect to proper time. An observer in such a coordinate system could easily mistake such a force to be that due to a uniform gravitational field.

As shown in the Appendix, the geodesic equation can be expressed in terms of the metric tensor $g_{\alpha\beta}$ in arbitrary coordinates and four-velocity u^{μ} as

$$\frac{d^2x^{\alpha}}{d\tau^2} + g^{\alpha\beta}u^{\mu}\frac{dg_{\mu\beta}}{d\tau} + g^{\alpha\beta}u_{\mu}u^{\mu}_{,\beta} = 0 \qquad (2)$$

where $d\tau$ represents an interval of proper time experienced by the observer, and commas denote differentiation. Applying Eq. (2) to the case of a uniformly accelerating observer, we note that a fixed coordinate system implies that $dg_{\mu\beta}/d\tau = 0$. Equation (2) can then be written as

$$\frac{d^2x^{\alpha}}{d\tau^2} + g^{\alpha\beta}u_0 u^0_{,\beta} = 0 \qquad (3)$$

in which u^0 is the time-component of the observer's four-velocity. Next, we note that in the local coordinate system of the accelerating observer, the components of the four-velocity can be expressed in the form [14-16]

$$u^{\mu} = \left(v^0\frac{dt}{d\tau}, 0, 0, 0\right) \qquad (4)$$

where dt represents a coordinate time interval in the accelerating system. Using these components in Eq. (3) then leads to

$$\frac{d^2x^{\alpha}}{d\tau^2} + g^{\alpha\beta}v_0 v^0\frac{dt}{d\tau}\partial_{\beta}\left(\frac{dt}{d\tau}\right) = 0 \qquad (5)$$

in which we now use ∂_{β} to represent the partial differential $\partial/\partial x_{\beta}$, and we have taken $\partial_{\beta}v^0 = 0$. To eliminate the factor $v_0 v^0$ in Eq. (5), we note that when the squared magnitude of the four-velocity $u_{\mu}u^{\mu} = c^2$ is written in terms of the components given in Eq. (4), it simplifies to $v_0 v^0 = c^2(d\tau/dt)^2$. Using this result, Eq. (5) can be written in the form

$$\frac{d^2x^{\alpha}}{d\tau^2} + c^2 g^{\alpha\beta}\partial_{\beta}\left[\ln\left(\frac{dt}{d\tau}\right)\right] = 0 \qquad (6)$$

Thus, the proper force experienced by the accelerating observer is

$$F^{\alpha} = -mc^2 g^{\alpha\beta}\partial_{\beta}\left[\ln\left(\frac{dt}{d\tau}\right)\right] \qquad (7)$$

where m is the observer's rest-mass.

When the acceleration is weak, as is the case with Newtonian forces, time dilation [17] $dt/d\tau$ assumes values very close to unity. The logarithm in Eq. (7) can then be approximated according to $\ln(dt/d\tau) \quad (dt/d\tau) - 1$. Carrying this out, and simplifying somewhat [18], we find that Eq. (7) can be written as

$$\mathbf{F} = E \quad \left(\frac{dt}{d\tau}\right) \qquad (8)$$

where E is the observer's rest mass-energy. This is the proper force experienced by the accelerating observer, written in three-dimensional notation.

According to Eq. (8), the proper force experienced by an accelerating observer arises simply out of a relationship between the observer's rest mass-energy and the local structure of time. In particular, whenever an observer resides at rest in a system of coordinates in which time dilation [17] $dt/d\tau$ varies as a function of that system's spatial coordinates, a force is observable in the direction of increasing dilation of time. Equation (8) therefore appears to answer the question posed in the Introduction. Space-time structure results in a force, on a body of rest mass-energy E, by the presence of a directional time-distortion. This implies that the inertial properties of matter have a local origin. Matter occupying a point at which a directional time-distortion exists, experiences a measurable force at that point. Thus, the interpretation of inertia set forth by Eq. (8) preserves causality, avoiding the need for instantaneous action at a distance phenomena.

Another thing to notice is that whenever time dilation $dt/d\tau$ assumes a constant value in Eq. (8), the force is zero. It then follows that if time were absolute, and thus incapable of distortion, inertial forces would not arise regardless of the quantity of matter in question. On this basis, therefore, it is evident that the local structure of time is the dynamic variable by which inertial forces originate.

3. Inertia of an Observer Accelerating Uniformly in Minkowski Space-Time

Equation (8) seems straightforward enough for the case of inertial forces experienced by a uniformly accelerating observer. But does Eq. (8) account for the inertial resistance of an accelerating observer from the point of view of stationary observers in flat, Minkowski space-time? To answer this, we use special relativity to derive the inertial resistance of an accelerating observer according to observers in Minkowski space-time.

Consider two observers, residing in Minkowski space-time, who conduct an experiment to determine the inertial resistance of an accelerating observer. As depicted by Figure 1, one observer, residing at origin **O'** of reference system **S'**, accelerates uniformly under the influence of a constant external force while a second observer remains stationary at origin **O** of system **S**. While accelerating, the moving observer focuses a light beam of proper frequency ν_0 on the stationary observer. Upon receiving the light beam, the stationary observer finds that the frequency is increasingly Doppler-shifted due to increasing relative velocity between the two observers:

$$\nu = \nu_0 \sqrt{\frac{1 - v/c}{1 + v/c}} \tag{9}$$

The stationary observer measures an initial frequency ν_i; and then at some later time during the experiment, takes a final measurement of the frequency ν_f. Assuming that the stationary observer has no direct means by which to measure the velocity of the moving observer, the stationary observer uses frequencies ν_i and ν_f to determine the inertial resistance of the moving observer.

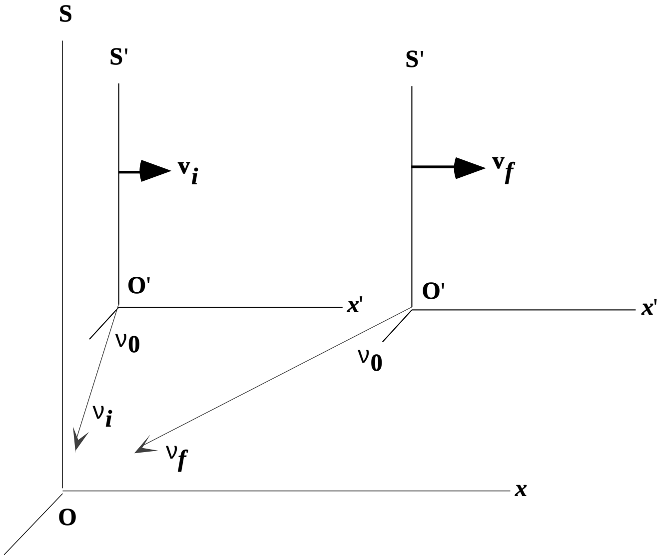

Figure 1. System S' accelerates under the influence of a constant external force oriented in the $+x$ direction of stationary system S. An observer residing at origin O' of co-moving system S' focuses a light beam of proper frequency ν_0 on a stationary observer residing at origin O of system S. The stationary observer measures frequencies ν_i and ν_f of light emitted by the moving observer at the instant when the relative velocity between systems S and S' is $\mathbf{v}_i$ and $\mathbf{v}_f$, respectively.

As the moving observer acquires greater velocity, time intervals in accelerating system **S'** appear increasingly dilated relative to those in stationary system **S**. The stationary observer can compute the dilation of time by solving Eq. (9) for v/c and then substituting the resulting expression into $dt/d\tau = 1/\sqrt{1 - v^2/c^2}$. Carrying this out leads directly to

$$\frac{dt}{d\tau} = 1 \Big/ \sqrt{1 - \left(\frac{\varepsilon - 1}{\varepsilon + 1}\right)^2} \tag{10}$$

where dt is a time interval in **S**, $d\tau$ is an interval of proper time in **S'**, and we have taken $\varepsilon = (\nu_0/\nu)^2$ [19]. After some algebraic manipulation, we find that Eq. (10) becomes

$$\frac{dt}{d\tau} = \frac{1}{2}\left(\frac{\nu_0}{\nu} + \frac{\nu}{\nu_0}\right) \tag{11}$$

This expression gives the dilation of time in the moving system in terms of the frequency observed in the stationary system. Using Eq. (11) for frequencies ν_i and ν_f, the stationary observer computes the time dilation by using

$$\left(\frac{dt}{d\tau}\right)_i = \frac{1}{2}\left(\frac{\nu_0}{\nu_i} + \frac{\nu_i}{\nu_0}\right) \tag{12a}$$

$$\left(\frac{dt}{d\tau}\right)_f = \frac{1}{2}\left(\frac{\nu_0}{\nu_f} + \frac{\nu_f}{\nu_0}\right) \tag{12b}$$

With these expressions in hand, the stationary observer can compute the moving observer's kinetic energy at the instant when the light was emitted:

$$T_i = mc^2\left[\left(\frac{dt}{d\tau}\right)_i - 1\right] \tag{13a}$$

$$T_f = mc^2\left[\left(\frac{dt}{d\tau}\right)_f - 1\right] \tag{13b}$$

in which we use $dt/d\tau$ in place of the usual factor $1/\sqrt{1 - v^2/c^2}$ [20]. Assuming that the force acting on the moving observer is conservative, the change in kinetic energy is equivalent to the work W performed in accelerating the moving observer. Using Eqs. (13), the work is then

$$W = T_f - T_i = E\left[\left(\frac{dt}{d\tau}\right)_f - \left(\frac{dt}{d\tau}\right)_i\right] \tag{14}$$

where E is the rest mass-energy of the moving observer. Thus, according to the stationary observer, the work performed in accelerating the moving observer is directly proportional to the change in the dilation of time in accelerating system **S'** relative to stationary system **S**.

To obtain the inertial resistance of the moving observer, we note that for the case of a constant external force, Eq. (14) can be expressed also as

$$W = F\Delta x \tag{15}$$

where F is the force and Δx is the distance over which the force acts to transform the moving observer's kinetic energy from T_i to T_f. Equating Eqs. (14) and (15), and taking the force to be oriented entirely in the $+x$ direction of system **S**, leads to

$$F_x = E\left\{\frac{1}{\Delta x}\left[\left(\frac{dt}{d\tau}\right)_f - \left(\frac{dt}{d\tau}\right)_i\right]\right\} \tag{16}$$

Upon taking the limit as Δx tends toward zero, we find that the force acting along the moving observer's world line is

$$F_x(x) = E\,\mathop{Lim}_{\Delta x \to 0}\left\{\frac{1}{\Delta x}\left[\left(\frac{dt}{d\tau}\right)_f - \left(\frac{dt}{d\tau}\right)_i\right]\right\} = E\frac{\partial}{\partial x}\left(\frac{dt}{d\tau}\right) \tag{17}$$

Noticing that this can be taken as the x-component of an arbitrarily directed three-force, the vector-force acting on the moving observer can be written as [21]

$$\mathbf{F} = E\left(\frac{dt}{d\tau}\right) \tag{18}$$

The resistance force of the moving observer is then obtained by simply taking the negative of Eq. (18):

$$\mathbf{F} = -E\left(\frac{dt}{d\tau}\right) \tag{19}$$

This is the inertial resistance of an accelerating observer according to stationary observers in Minkowski space-time.

Equation (19) is identical to Eq. (8), obtained on the basis of general relativity for the case of Newtonian forces. However, the fact that Eq. (19) applies to Minkowski space-time leads to a slightly different interpretation. As a given body accelerates under the action of an external force, proper time in the comoving reference frame of the body becomes increasingly distorted relative to time in the inertial frames of Minkowski space-time. Stationary observers that study such a body find that the magnitude of time distortion changes as the body accelerates or decelerates through Minkowski space-time. According to Eq. (19), the changing time distortion gives rise to a second force on the body that is equal, but oppositely directed to the externally applied force. The second force is the inertial resistance of the body, measurable in Minkowski space-time. Thus, as in Sec. II, we find that the relationship between time and energy plays a fundamental role in the inertial properties of matter.

Another important point to recognize is that the presence of the gradient operator in Eq. (19) makes it clear that if time were absolute, matter would not exhibit inertial properties. More specifically, if time were absolute in the classical sense, all observers would measure identical time intervals. One would then have $dt / d\tau = 1$ for all observers regardless of their relative motions. According to Eq. (19), the inertial resistance of these observers would then be zero. This implies that time distortion is the dynamic variable by which inertial forces originate.

4. Conclusions

We have shown that special and general relativity provide a clear and simple statement regarding the origin of inertia. We began by pointing out that according to general relativity, inertial and gravitational forces arise directly out of space-time structure [8-13]. This implies that an adequate description of inertia cannot be obtained on the basis of inertial mass alone, but must also take the behavior of space-time into account. General relativity was then used to derive an expression for the proper force experienced by an observer accelerating uniformly under the influence of a constant external force. Next, we used special relativity to derive the inertial resistance of an observer accelerating under the influence of a constant external force in flat, Minkowski space-time. Both analyses led to an expression for the force implying that the origin of inertia is a relationship between the energy content of matter and the local structure of time. In both cases, we found that the expression for the force implies that if time were absolute, inertial forces would not arise regardless of the quantity of matter in question. This led us to state that time distortion is the dynamic variable by which inertial forces originate. On this basis, therefore, we conclude that the origin of inertia is the relativistic nature of time.

We should also point out that the conclusions drawn herein in no way undermine the zero-point-field (ZPF) proposal [1-4] discussed in the Introduction. It is clear that space-time structure not only dictates the motion of material bodies but also the behavior of zero-point fluctuations. According to the ZPF theory, it is precisely this behavior that induces the inertial properties of matter. If this is indeed the case, then the ZPF theory provides a physical description of the process by which space-time structure generates observable forces. However, since space-time structure dictates the behavior of zero-point fluctuations, it must be kept in mind that space-time is necessarily more fundamental than the zero-point field.

Appendix

To derive the proper force experienced by an observer in an arbitrary coordinate system, it is helpful to have a form of the geodesic equation expressed in terms of the observer's four-velocity u^μ. The geodesic equation in arbitrary coordinates is

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{A1}$$

where the Christoffel symbol is given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\mu\beta,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right) \tag{A2}$$

in which $g_{\alpha\beta}$ is the metric tensor in arbitrary coordinates, and commas denote partial differentiation. Using the Christoffel symbol, the geodesic equation can be written in terms of the metric tensor and four-velocity as

$$\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}g^{\alpha\beta}\left(g_{\mu\beta,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right)u^\mu u^\nu = 0 \tag{A3}$$

We now notice an immediate simplification: the product $u^\mu u^\nu$ is symmetric in the indices μ and ν, and the derivatives of the metric tensor in the second and third terms, $g_{\mu\beta,\nu}$ and $g_{\beta\nu,\mu}$, are also symmetric in the indices μ and ν. Thus, the second and third terms in Eq. (A3) add:

$$\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2} g^{\alpha\beta}\left(2g_{\mu\beta,\nu}u^\mu u^\nu - g_{\mu\nu,\beta}u^\mu u^\nu\right) = 0 \tag{A4}$$

To simplify the last term of Eq. (A4), we differentiate $g_{\mu\nu}u^\mu u^\nu = c^2$ with respect to x^β. Carrying this out leads to

$$g_{\mu\nu,\beta}u^\mu u^\nu + u_\nu u^\nu_{,\beta} + u_\mu u^\mu_{,\beta} = 0 \tag{A5}$$

where we have used metric tensor $g_{\mu\nu}$ to lower the index of the four-velocity in each of the last two terms. Equation (A5) simplifies further upon noticing that the last two terms add when summed over dummy indices μ and ν. Carrying this out, and simplifying somewhat, gives

$$g_{\mu\nu,\beta}u^\mu u^\nu = -2u_\mu u^\mu_{,\beta} \tag{A6}$$

To simplify the second term in Eq. (A4), we simply note that [22]

$$g_{\mu\beta,\nu}u^\mu u^\nu = u^\mu \frac{\partial g_{\mu\beta}}{\partial x^\nu}\frac{dx^\nu}{d\tau} = u^\mu \frac{dg_{\mu\beta}}{d\tau} \tag{A7}$$

Upon substituting Eqs. (A7) and (A6) into Eq. (A4), we arrive at

$$\frac{d^2x^\alpha}{d\tau^2} + g^{\alpha\beta}u^\mu \frac{dg_{\mu\beta}}{d\tau} + g^{\alpha\beta}u_\mu u^\mu_{,\beta} = 0 \tag{A8}$$

These are the equations of motion of an observer moving in arbitrary coordinates, and are equivalent to the original form of the geodesic equation.

Notes and References

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[5] See, for example, Max Born, **Einstein's Theory of Relativity**, p. 283 (Dover, New York, 1965).

[6] Max Born, Ref. [5], p. 286.

[7] H. Weyl, **Space-Time-Matter**, 4th ed., p.202 (Dover, New York, 1952).

[8] A. Einstein, **The Meaning of Relativity**, *Including the Relativistic Theory of the Non-Symmetric Field*, 5th ed., pp. 57-58 (Princeton, New Jersey, 1988).

[9] Max Born, Ref. [5], pp. 313-317.

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[11] B. F. Schutz, **A First Course in General Relativity**, p.122 (Cambridge, New York, 1990).

[12] H. Ohanian and R. Ruffini, **Gravitation and Spacetime**, 2nd ed., pp. 53-54 (Norton, New York, 1994).

[13] I. R. Kenyon, **General Relativity**, p. 10 (Oxford, New York, 1990).

[14] Here, we note that were the observer at rest in an inertial frame, the four-velocity would assume the form $u^\mu = (v^0, 0, 0, 0)$, where $dt/d\tau = 1$.

[15] See B. F. Schutz, Ref. [11], p. 76; H. Ohanian and R. Ruffini, Ref. [12], p. 76; and I. R. Kenyon, Ref. [13], p. 7.

[16] A. O. Barut, **Electrodynamics and Classical Theory of Fields and Particles**, p. 48 (Dover, New York, 1980).

[17] Here we use the word "dilation" in reference to all instances in which space-time structure leads to a dilation of time; and for convenience sake, we use the term "time dilation" in reference to the factor $dt/d\tau$. See, for example, P. G. Bergmann, Ref. [10], p. 44.

[18] As a further approximation, we take the inverse metric tensor to be $g^{\alpha\beta} \quad \eta^{\alpha\beta} + h^{\alpha\beta}$, and we neglect terms of order higher than $dt/d\tau$. Here, $\eta^{\alpha\beta}$ is the inverse metric tensor of Minkowski space-time with signature (+,-,-,-), and $h^{\alpha\beta}$ is the inverse metric tensor of a Newtonian acceleration field.

[19] Here, we treat the electromagnetic oscillations of the light beam as a clock that is observable in both the stationary and moving frames. See, for example, R. Resnick, **Introduction to Special Relativity**, pp. 90-91 (Wiley, New York, 1968); and C. Lanczos, **The Variational Principles of Mechanics**, 4th ed., p. 339 (Dover, New York, 1968).

[20] To exemplify the relativity of time, we choose to express formulae in terms of $dt/d\tau$ rather than the usual factor $1/\sqrt{1 - v^2/c^2}$. See, for example, P. G. Bergmann, Ref. [10]; H. Minkowski, "Space And Time", p. 87 in **Einstein, The Principle of Relativity** (Dover, New York, 1952); and C. W. Misner, K. S. Thorne, and J. A. Wheeler, **Gravitation**, p. 50 (Freeman, New York, 1973).

[21] Although we use a somewhat lengthy derivation, a more straightforward approach is to use the canonical equation of motion $dp_i / dt = -\partial H / \partial x^i$ in conjunction with the relativistic Hamiltonian for a moving observer, given by $H = mc^2 dt/d\tau$. See, for example, J. B. Marion and S. T. Thornton, **Classical Dynamics of Particles & Systems**, 3rd ed., pp. 528, 538-539 (Academic, New York, 1988).

[22] See B. F. Schutz, Ref. [11], p. 186; I. R. Kenyon, Ref. [13], p. 187.