

Time-dependent electric field outside a charged sphere

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There can be no doubt that solving the wave equation in spherical coordinates is one of the central problems in Electrodynamics. Unfortunately, the steps required to solve the wave equation are not typically shown. With this in mind, let us consider the case in which a spherical conductor has a surface charge that is made to vary sinusoidally over time. The electric potential in spherical coordinates, $\phi(r, \theta, \varphi, t)$, is expressed by use of the usual wave equation:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0$$

where we assume a general solution of the form

$$\phi(r, \theta, \varphi, t) = \sum_k \psi_k(r, \theta, \varphi) e^{-ickt}$$

Separation of variables leads to the familiar Helmholtz equation:

$$\nabla^2 \psi_k + k^2 \psi_k = 0$$

The general solution to the Helmholtz equation can be immediately written as

$$\psi_k(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(A_{lm}^{(1)} h_l^{(1)}(kr) + A_{lm}^{(2)} h_l^{(2)}(kr) \right) Y_{lm}(\theta, \varphi)$$

where the spherical Hankel functions are expressible in terms of spherical Bessel functions:

$$h_l^{(1)}(kr) = j_l(kr) + in_l(kr)$$

$$h_l^{(2)}(kr) = j_l(kr) - in_l(kr)$$

Upon considering the case in which waves propagate from the conductor out to infinity, it is straightforward to see that $A_{lm}^{(2)} = 0$. The general solution to the Helmholtz equation then simplifies to

$$\psi_k(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l A_{lm}^{(1)} h_l^{(1)}(kr) Y_{lm}(\theta, \varphi)$$

The general solution to the wave equation is then expressible as

$$\phi(r, \theta, \varphi, t) = \sum_{klm} A_{lm}^{(1)} h_l^{(1)}(kr) Y_{lm}(\theta, \varphi) e^{-ickt}$$

We can refer to this expression as the first solution to the wave equation. Next, we must solve for $A_{lm}^{(1)}$. Let's consider the potential ψ_k to be a function of $(r_c, \theta_c, \varphi_c)$ on a boundary surface at a time $t = 0$. The potential is then

$$\psi_k(r_c, \theta_c, \varphi_c) = \sum_{lm} A_{lm}^{(1)} h_l^{(1)}(kr_c) Y_{lm}(\theta_c, \varphi_c)$$

Upon multiplying both sides by $Y_{l'm'}^*(\theta_c, \varphi_c)$, integrating over the solid angle Ω , and using the orthogonality condition for spherical harmonics, we arrive at

$$\begin{aligned} \int d\Omega \psi_k(r_c, \theta_c, \varphi_c) Y_{l'm'}^*(\theta_c, \varphi_c) &= \sum_{lm} A_{lm}^{(1)} h_l^{(1)}(kr_c) \int d\Omega Y_{lm}(\theta_c, \varphi_c) Y_{l'm'}^*(\theta_c, \varphi_c) \\ &= A_{l'm'}^{(1)} h_{l'}^{(1)}(kr_c) \end{aligned}$$

Solving for $A_{lm}^{(1)}$ and dropping the primes on the subscripts leads to

$$A_{lm}^{(1)} = \frac{1}{h_l^{(1)}(kr_c)} \int d\Omega \psi_k(r_c, \theta_c, \varphi_c) Y_{lm}^*(\theta_c, \varphi_c)$$

Substituting this expression for $A_{lm}^{(1)}$ into the first solution to the wave equation gives

$$\phi(r, \theta, \varphi, t) = \sum_{klm} \frac{h_l^{(1)}(kr)}{h_l^{(1)}(kr_c)} Y_{lm}(\theta, \varphi) e^{-ickt} \int d\Omega \psi_k(r_c, \theta_c, \varphi_c) Y_{lm}^*(\theta_c, \varphi_c)$$

With a general solution now in hand, let's consider the case of a uniformly charged sphere. For the case of spherical symmetry, we have $l, m = 0$. Moreover, when the surface potential of the sphere is uniformly distributed, we may put $\psi_k(r_c, \theta_c, \varphi_c) = \psi_k(a)$, where a is the radius of the sphere. Under these conditions, the general solution becomes

$$\phi(r, t) = \frac{ae^{ik(r-a)}}{4\pi r} e^{-ickt} \psi_k(a) \int d\Omega$$

where the lack of angular dependence on $\psi_k(a)$ allows it to be moved outside the integral, and where the following functions have been used:

$$h_0^{(1)}(kr) = \frac{e^{ikr}}{ikr}$$

$$Y_{00}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$$

For a uniformly charged sphere, we may put

$$\psi_k(a) = \frac{q}{a}$$

where q is the total charge on the sphere at $t = 0$ and a is the radius of sphere, as stated above. When $l, m = 0$, the integral over the solid angle Ω takes on the particularly simple form:

$$\int d\Omega = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta = 4\pi$$

The expression for $\phi(r, t)$ outside the charged sphere then becomes

$$\phi(r, t) = \frac{q}{r} e^{-ick\left(t - \frac{(r-a)}{c}\right)}$$

This expression can also be expressed in a more familiar form by using the relation $\omega = ck$:

$$\phi(r, t) = \frac{q}{r} e^{i[k(r-a) - \omega t]}$$

Now that we have an expression for the potential outside the sphere, let's find the resulting time-dependent electric field outside the sphere. We begin with the familiar equation

$$\mathbf{E} = -\nabla\phi$$

Since the potential is independent of (θ, φ) , the electric field is

$$\mathbf{E}(r, t) = \hat{\mathbf{r}} \frac{q}{r} \left(\frac{1}{r} - \frac{i\omega}{c} \right) e^{-ick\left(t - \frac{(r-a)}{c}\right)}$$

Restricting the electric field to just the real portion leads to

$$\mathbf{E} = \hat{\mathbf{r}} \frac{q}{r^2} \cos\left(ck\left(t - \frac{(r-a)}{c}\right)\right) - \hat{\mathbf{r}} \frac{\omega q}{rc} \sin\left(ck\left(t - \frac{(r-a)}{c}\right)\right)$$

Upon rearranging a bit, and using $\omega = ck$ leads to a familiar expression for the electric field:

$$\mathbf{E} = \hat{\mathbf{r}} \frac{q}{r^2} \cos(k(r-a) - \omega t) + \hat{\mathbf{r}} \frac{\omega q}{rc} \sin(k(r-a) - \omega t)$$

As expected, when the charge is varied on the surface of the sphere, the electric field varies accordingly. The electric field variations propagate outwardly from the surface of the sphere to infinity in the form of spherical waves.

Another interesting situation is the case of incoming waves—that is, spherical waves traveling from infinity toward the surface of the sphere. When considering this case, we must refer to the second solution to the wave equation, given above in the form:

$$\phi(r, \theta, \varphi, t) = \sum_{klm} A_{lm}^{(2)} h_l^{(2)}(kr) Y_{lm}(\theta, \varphi) e^{-ickt}$$

Using the techniques demonstrated above to determine $A_{lm}^{(2)}$ leads to

$$A_{lm}^{(2)} = \frac{1}{h_l^{(2)}(kr_c)} \int d\Omega \psi_k(r_c, \theta_c, \varphi_c) Y_{lm}^*(\theta_c, \varphi_c)$$

Substituting this expression for $A_{lm}^{(2)}$ into the second solution to the wave equation gives

$$\phi(r, \theta, \varphi, t) = \sum_{klm} \frac{h_l^{(2)}(kr)}{h_l^{(2)}(kr_c)} Y_{lm}(\theta, \varphi) e^{-ickt} \int d\Omega \psi_k(r_c, \theta_c, \varphi_c) Y_{lm}^*(\theta_c, \varphi_c)$$

When we choose the spherically symmetric case, as above, the spherical Hankel function of the second kind, $h_l^{(2)}(kr)$, assumes the form

$$h_0^{(2)}(kr) = \frac{ie^{-ikr}}{kr}$$

Using this expression and the same boundary conditions given above for the spherical conductor leads directly to

$$\phi(r, t) = \frac{q}{r} e^{-i[k(r-a)+\omega t]}$$

where we have also used the relation $\omega = ck$. This expression is the electric potential due to incoming waves. Adding the incoming and outgoing potentials, and simplifying a bit, gives

$$\phi(r, t) = \frac{2q}{r} e^{-i\omega t} \cos(k(r-a))$$

As before, the electric field can be straightforwardly determined by using

$$\mathbf{E} = -\nabla\phi$$

Carrying out the gradient and retaining the real portion of the resulting electric field yields

$$\mathbf{E} = \hat{\mathbf{r}} \frac{2q}{r^2} e^{-i\omega t} \cos(k(r-a)) + \hat{\mathbf{r}} \frac{2\omega q}{rc} e^{-i\omega t} \sin(k(r-a))$$

This expression suggests that the electric field assumes the form of standing spherical waves outside the charged sphere. ■